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PEB

PHYSICAL OPTICS,

PART II.

THE CORPUSCULAR THEORY OF LIGHT:

DISCUSSED MATHEMATICALLY.

BY

RICHARD POTTER, A.M. F.C.P.S.

LATE FELLOW OF QUEENS' COLLEGE, CAMBRIDGE; LICENTIATE OF THE ROYAL COLLEGE
OF PHYSICIANS, LONDON; HONORARY MEMBER OF THE LITERARY AND
PHILOSOPHICAL SOCIETY OF ST. ANDREW'S;
PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY
IN UNIVERSITY COLLEGE, LONDON.

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PREFACE.

THE proposal of a vibratory theory of light by Des Cartes was of little value until Huygens advanced to the undulatory theory, which he left nearly as it is at present received, but with the hypothesis of transversal vibrations added by Dr Young and M. Fresnel to account for the polarization of light. Newton, having adopted a mixed vibratory and corpuscular theory, was by his doctrine of the polarization of light enabled to give a reason for phenomena observed in the double refraction of calc spar which Huygens could not reconcile with an undulatory theory. Newton was, however, unfortunate in his hypothesis of 'fits of easy reflexion and transmission,' formed to account for the occurrence of periodical colours; and his great authority during a century prevented the doctrine of interference, which had been advanced by Dr Hooke, from being accepted as the true explanation of the several cases investigated.

The Newtonian theory numbers, however, amongst its disciples the great names of Laplace and Malus, and the latter must ever rank as a chief leader in the advance of Physical Optics, by his mathematical investigations and his discovery of the polarization of light by reflexion at transparent surfaces.

The revival of the undulatory theory with the doctrine of interference, at the beginning of the present century, is due to Dr Young, and its reception by mathematicians to M. Fresnel's analytical researches, which promised, when fully developed, to comprehend the discoveries of himself, of M. Arago, M. Biot, Dr Brewster, M. Fraunhofer and others.

The expressing in mathematical formulæ the wonderful and beautiful phenomena shown in the interference of ordinary and polarized light, acted like an enchantment upon the mathematicians, and their glory and pride was to develop so surprising a theory. The inertia of the previous century came now into effect again in favour of the undulatory theory of light, and the investigations which professed to confirm or advance it were extolled, whilst those which militated against it were rejected or met by an assertion that at the utmost some subsidiary hypothesis might be needed.

In the experiment with the two mirrors slightly inclined, M. Arago had stated that the central interference-bar was always a bright one, in accordance with the undulatory theory; but it was found frequently to be seen a dark one, contrary to that theory, and means were sought to secure for exhibition a bright central bar. This was first accomplished by passing the light at the luminous point through a minute aperture in a thin opaque plate so that the light was in a state of interference by diffraction before falling upon the two mirrors, and the interference produced by them was secondary, but it was exhibited as primary interference. Another ingenious method to produce the same result is by forming the luminous point or line with lenses, so as to have interference near a caustic, before the light falls upon the two mirrors. In the case of the so-called spurious rainbows the measures of the bright bars did not accord well with the theory, whilst those of the dark bars did so; after full consultation, therefore, the former were suppressed, and the latter published.

With the reiteration of blunders and the management described above, together with bold assertions of the most complete agreement of the facts of experiment with conclusions from the undulatory theory, it was asserted to be as certainly true as the theory of gravitation.

With such advocacy it was not likely that the author of the present treatise would find companions in investigating critical points where the undulatory theory fails, and he has had the field nearly clear to himself for thirty years, until he had completed the experimental and mathematical investigations discussed in the Two Parts of the Work. The present Second Part was finished before the First Part was printed; but on account of the delay in getting it published he was enabled to revise, correct, and expand the manuscript to some small extent. The author is far from considering that he has done more than commence the mathematical discussion of the corpuscular theory of light; but trusts that Physical Optics, recalled from one of its wanderings, may by-and-by take a straight course of progress, and that his long perseverance against dogmatic error will not be considered as lost labour by future investigators.

LONDON, 1859.

CONTENTS.

PAGE

INTRODUCTION 1

Failures of the undulatory theory of light—it must be abandoned—corpuscular theory—luminiferous corpuscles in sheets with given intervals—have poles, axes, and equators—polarization of light, 2; how arising—anatropy.

CHAPTER I.

ON THE REFLEXION AND REFRACTION OF LIGHT . . . 3

Velocity of a luminiferous molecule after reflexion, 4; of a refracted molecule, 5; impulsive forces at the refracting surface—coefficients investigated, 5; forms of the luminiferous surfaces in uniaxal crystals, 6; ordinary and extraordinary rays, 7; sphere and spheroid, 8; retardation in uniaxal crystals—how expressed in positive and negative crystals, 10; relative directions of the two rays—when the refracting surface is a principal plane, 11; critical angles in calc-spar, 13; experimental verification, 14; quartz a peculiar uniaxal crystal—most general case of double refraction—biaxal crystals, 15; inner and outer sheets of the luminiferous surface, 16; the rays belonging to each, 17; the equations of the luminiferous surfaces in biaxal crystals, 19; angle between the optic axes, 21; angular separation of the two rays—radius of the eyelet-hole, *ib.*; law of the retardation in biaxal crystals, 23; the refractive indices of arragonite, sulphate of barytes and topaz—the optic axes in arragonite, 25.

CHAPTER II.

ON THE POLARIZATION OF LIGHT AT TRANSPARENT SURFACES . 26

Axes of the molecules are in the luminiferous surfaces, 26; polarization of a beam after reflexion at a transparent surface, 27; Brewster's law, 28; polarization of the refracted ray, 29; of the emergent ray, 30; of the internally reflected ray, 31; of the emergent ray after internal reflexion, 32; polarization of totally reflected rays, 33; rhombs to produce circularly polarized light, 34.

CHAPTER III.

ON THE INTENSITY OF LIGHT REFLECTED AT POLISHED SURFACES 35

Results of photometry—law of the intensity of the reflexion at transparent surfaces, 35; the constants for crown, plate and flint glass, 36; connected with specific heats—intensity of the reflexion by metals, 37; intensity of the reflexion of polarized light, 38.

PAGE

CHAPTER IV.

ON THE INTERFERENCE OF ORDINARY LIGHT . . 39

Value of the luminiferous interval λ determined by experiment, 40; experiment of the two mirrors slightly inclined, *ib.*; investigation of the formula for the intensity in interfering beams from Newton's rings, 42; exponential formula—value of the constant ν, 44; Arago's experiment considered, 45; interference by two plane mirrors at any point, 46; the velocity of light in glass from experiment, 48; radii of Newton's rings, 49; when the light is obliquely incident, 51; interference near the shadow of a body, or diffraction, 52; law of the fringes, 53; interference within the shadow of a narrow body, 54; the coloured bars in hyperbolic arcs, 55; interference of light through a small aperture which is a parallelogram, 56; results compared with Fraunhofer's laws, 57; investigations for different forms of apertures—narrow angular aperture, 59; small circular aperture, 60; a grating—the normal or interference spectrum, 61; the luminous point not very distant, 62; diffraction with a telescope, 63; effect of the object-glass, 64; interference near a caustic curve, *ib.*; forms of luminiferous surfaces after reflexion by a spherical mirror, 65; equations investigated, 66; law of the distances of the rings, 67; the rainbow, 69; the asymptote of the caustic, 70; the caustic curve investigated, 72; results for the supernumerary rainbows, 73.

CHAPTER V.

ON THE INTERFERENCE OF POLARIZED LIGHT . . 7

Interference formula for polarized light, 75; interference in the light transmitted by a plate of a uniaxal crystal, 77; forms of the rings and crosses, 78; the double cross and segments of rings, 80; the discussion of circularly polarized light, 81; of elliptically polarized light, 83; interference when circularly polarized light passes through a plate of a uniaxal crystal, 84; the cross and segments of rings, 85; the light circularly polarized and circularly analyzed, 87; the interference by quartz crystal discussed, 89; rotatory polarization examined, 91; laws of the rings discussed, 93; circularly polarized light transmitted by quartz—the law of the spirals, 95; interference by plates of biaxal crystals investigated, 96; lemniscates, 97; forms of the brushes, 98; the incident light circularly polarized, 100; interference by a cylindrical rod of unannealed glass; irregular forms of glass plates and irregular crystals, 101.

PHYSICAL OPTICS.

THE SCIENCE OF THE NATURE AND PROPERTIES OF LIGHT.

PART II.

INTRODUCTION.

MANY cases were pointed out in Part I. where the undulatory theory fails to represent the phenomena which are observed; and these were not such, that modifications of the subsidiary hypotheses would produce conformable results, and bring that theory to represent them, but were at variance with the essential properties of waves. Of these the darkness at caustic curves is observed, in place of brightness, which that theory requires; the enormous discrepancies of the intensities in Newton's transmitted rings compared with those given by that theory; the glaring failure of the expression for the light reflected at a perpendicular incidence from transparent surfaces; the failure of Fresnel's expression for the wave surface in biaxal crystals, are examples: and many other instances may be brought forward.

The undulatory theory therefore being abandoned, we must recur to a corpuscular theory, with the principle of periodicity as an essential property of light; or the luminiferous corpuscles must be considered as flying off in *surfaces*, *sheets*, or *shells* from luminous points, with intervals (λ) which are constant for the same colour of the solar spectrum, but which vary from colour to colour, and have the values in the table, page 70, Part I.

The property of polarization of beams of light is to be considered as arising from the luminiferous corpuscles having different properties on different sides, or having poles, axes, and equators, similar to what are recognized in the particles ponderable matter, and a beam is called a polarized beam whe the axes of the luminiferous corpuscles are all parallel. It wi be found in the chapter on polarization that the axes must b considered to be in the luminiferous surfaces, and therefore the planes of the equators at right angles to them; and that the plane of polarization of a beam of light is the plane to which the planes of the equators of all the corpuscles are parallel.

Again, the axes of the corpuscles may be similarly situated, but the poles reversed in two interfering beams; and in this way the property which has been termed anatropy may arise, in some cases.

CHAPTER I.

ON THE REFLEXION AND REFRACTION OF LIGHT.

THE principal phenomena of the reflexion and refraction of light were explained in Chapter I. of Part I. of this treatise. We saw there that the reflexion of light might be compared with the rebounding of a perfectly elastic ball after being thrown against a hard surface; the component of the velocity of projection parallel to the surface being the same after as before impact, whilst the component perpendicular to it was first destroyed and then by virtue of the elasticity regenerated in the opposite direction. The analogy here ceases, however, for when light is incident upon a reflecting surface part only is reflected, a small part is dispersed, and the remainder enters the substance of the body.

The general properties of light in the cases of reflexion and refraction having been discussed in Part I., we have need only to reconsider those properties which can be investigated mathematically from physical data, and thus to find the causes of many of these properties.

ART. 1. PROP. *In the reflexion of light the velocity of the reflected ray equals that of the incident ray.*

For a luminiferous molecule which is reflected, we suppose the component of the velocity before incidence parallel to the reflecting surface, equals that after reflexion; and the component perpendicular to the surface equals that after reflexion, but in the opposite direction. Let BAC in figure 1 be a reflecting surface, SA a ray incident at A, AR the reflected ray, AN the normal at A, and let SAN the angle of incidence equal i, RAN the angle of reflexion equal i'.

Let v be the velocity of a luminiferous molecule in the ray SA, u the velocity of it in the reflected ray AR.

Then from the above relation of the parallel and perpendicular components we have

$$u \sin i' = v \sin i,$$

$$u \cos i' = -v \cos i;$$

$$\therefore \tan i' = -\tan i, \text{ and } i' = -i.$$

The plane of reflexion is also, from the conditions, the plane of incidence,

$$\text{also} \quad u^2 \sin^2 i' + u^2 \cos^2 i' = v^2 \sin^2 i + v^2 \cos^2 i,$$

$$\text{or } u^2 = v^2.$$

COR. The converse of the above proposition evidently holds good, when the velocity after reflexion equals that before, and the angle of incidence equals the angle of reflexion.

The light which undergoes refraction in a transparent medium is subject to impulsive forces at the surface, which act through an inappreciable space only, and then within the medium the path of the ray is a straight line.

Those media which have ordinary refraction, act with the same impulsive forces at their surfaces upon the molecules of light polarized in any planes; but double refracting substances first analyze the incident beam into two beams polarized in planes at right angles to each other, and then exert impulsive forces upon the luminiferous molecules of these two beams, which differ. Cases of the variation of these impulsive forces in nearly every imaginable manner are already known, as may be concluded from the description of the double refraction and interference results in Part I.

In order to find the laws of the action of these impulsive forces, we have to consider as given the fundamental properties of the ordinary refraction of transparent bodies; namely, that the sine of the angle of refraction bears a constant ratio to the sine of the angle of incidence, and that the velocity of the luminiferous molecules in the refracting medium bears the same ratio to the velocity in a vacuum, which the sine of the angle of refraction bears to the sine of the angle of incidence*. That is, if i be the angle of incidence, i' the angle of refraction, as in the next proposition, μ the refractive index; v the velocity of the incident molecule, u that of the refracted molecule; we have

$$\sin i = \mu \sin i',$$

$$v = \mu u.$$

With these data we require to find how the impulsive forces act in directions parallel and perpendicular to the refracting surface at the point of incidence.

ART. 2. PROP. *To investigate the coefficients which determine the velocities of an ordinarily refracted luminiferous molecule, parallel and perpendicular to the refracting surface.*

Let Ox, Oy, Oz be the rectangular axes of co-ordinates; and let the plane of xy be in the refracting surface of the medium. Let SO be a ray incident in the plane of xz, which will not affect the generality of the discussion, since the refraction is supposed to be ordinary refraction; let the angle of incidence SOz' equal i; let OR be the refracted ray, and the angle $ROz = i'$.

Let v be the velocity of the molecules of light in the incident ray,

u .... refracted ray,

and μ the refractive index.

Then we have the data, $\sin i = \mu \sin i'$, and $\mu u = v$.

Let l, m, n be the coefficients due to the impulsive forces, which multiplied into the velocity of the luminiferous molecules of the incident ray parallel to the co-ordinate axes Ox, Oy, Oz, give the velocities of the molecules in the refracted ray respectively, parallel to those axes; or

let $u \sin i' = l \,.\, v \sin i$..........................(1),

$u \cos i' = n \,.\, v \cos i$..........................(2),

and in ordinary refraction $m = l$;

---

* See Part I. page 9.

from (1) $$l = \frac{u \sin i'}{v \sin i} = \frac{u \sin i'}{\mu u \,.\, \mu \sin i'}$$

$$= \frac{1}{\mu^2};$$

from (2) $$n = \frac{u \cos i'}{v \cos i},$$

and $$n^2 = \frac{u^2 (1 - \sin^2 i')}{\mu^2 u^2 \cos^2 i}$$

$$= \frac{\mu^2 - \sin^2 i}{\mu^4 \cos^2 i}$$

$$= \frac{1}{\mu^4} + \frac{\mu^2 - 1}{\mu^4} \sec^2 i.$$

This value of n^2, with $l^2 = \frac{1}{\mu^4}$, and $m^2 = \frac{1}{\mu^4}$, are the squares of the coefficients required.

Art. 3. Prop. *To find the forms of the luminiferous surface in a plate of a uniaxal crystal cut perpendicularly to the axis, when the ordinary and extraordinary rays traverse the axis with the same velocity.*

Let the plane of xy be the plane of the surface of the plate, and let Oz be the axis of the crystal passing through the point of incidence O, of a ray SO.

The refraction being symmetrical with respect to the axis Oz, the coefficients for the velocities parallel to the plane of xy are the same in all directions around Oz, and we may take the plane of xz for the plane of incidence.

Let the angle $SOz' = i$, the angle of incidence; and OR, OR' be the two refracted rays, of which OR is the ordinary ray and polarized in the plane of incidence and refraction, which is a principal plane of the crystal*; and OR' the extraordinary ray polarized perpendicularly to the principal plane.

---

* See Part I. page 16.

Let OR make the angle of refraction $ROz = i'$ with Oz,
OR' .... $R'Oz = i''$ ....

Now OR being the ordinary ray, the coefficients l, m, n will be as in the last Article, or

$$l^2 = m^2 = \frac{1}{\mu^4},$$

$$n^2 = \frac{1}{\mu^4} + \frac{\mu^2 - 1}{\mu^4} \sec^2 i,$$

and OR' being the extraordinary ray, let the coefficients be l', m', n' with the velocities of the two rays equal in Oz, but unequal in other directions, and

$$l'^2 = m'^2 = \frac{1}{\mu'^4},$$

$$n'^2 = \frac{1}{\mu^2 . \mu'^2} + \frac{\mu'^2 - 1}{\mu^2 . \mu'^2} \sec^2 i.$$

Let v be the velocity of the luminiferous molecule in the incident ray, u its velocity in the ordinarily refracted ray, and u' its velocity in the extraordinary one; we have

$$u^2 \sin^2 i' = l^2 v^2 \sin^2 i = \frac{v^2}{\mu^4} \sin^2 i,$$

$$u^2 \cos^2 i' = n^2 v^2 \cos^2 i$$

$$= \frac{v^2}{\mu^4} (\mu^2 - 1 + \cos^2 i);$$

$$\therefore \sin^2 i = \frac{\mu^4 u^2 \sin^2 i'}{v^2},$$

$$\cos^2 i = \frac{\mu^4 u^2 \cos^2 i'}{v^2} - \mu^2 + 1,$$

and since $$\sin^2 i + \cos^2 i = \frac{\mu^4 u^2}{v^2} (\sin^2 i' + \cos^2 i') - \mu^2 + 1,$$

or $$\mu^2 u^2 = v^2,$$

which is identical with one of the fundamental data, and if t be the time of traversing any distance in OR, and O is the focus to which a pencil of incident rays converges, then ut is the radius vector of the luminiferous surface within the plate, and

$$ut = \frac{v}{\mu} t,$$

which varies only with t, and is therefore the radius of a sphere with center O, as we conclude also from *à priori* considerations; and the point O may be taken as a luminous origin upon the surface.

Again, for the luminiferous molecule of the extraordinary ray,

$$u'^2 \sin^2 i'' = l'^2 v^2 \sin^2 i$$

$$= \frac{v^2 \sin^2 i}{\mu'^4},$$

$$u'^2 \cos^2 i'' = n'^2 v^2 \cos^2 i$$

$$= \frac{v^2 \cos^2 i}{\mu^2 . \mu'^2} \{1 + (\mu'^2 - 1) \sec^2 i\},$$

and substituting as before,

$$\sin^2 i + \cos^2 i = 1 = \frac{u'^2}{v^2} (\mu'^4 \sin^2 i'' + \mu^2 . \mu'^2 \cos^2 i'') - \mu'^2 + 1,$$

whence $$u'^2 = \frac{v^2}{\mu'^2 \sin^2 i'' + \mu^2 \cos^2 i''},$$

or $$u'^2 . t^2 = \frac{\frac{v^2}{\mu'^2} . t^2}{1 + \left(\frac{\mu^2}{\mu'^2} - 1\right) \cos^2 i''},$$

and thus multiplying both sides by t^2, we have the equation of an ellipse to the center, between the radius vector ut and the angular ordinate i'', which is measured from the major or minor axis of the ellipse, as $\left(\frac{\mu^2}{\mu'^2} - 1\right)$ is negative or positive respectively.

For if

$AO = a =$ semi-major axis of the ellipse in figs. (4) and (5),

$BO = b =$ semi-minor ....

$PO = \rho =$ radius vector, and angle $POB = \theta$,

also $POA = \phi$;

then by conic sections, we have

$$\rho^2 = \frac{a^2}{1 + \left(\frac{a^2}{b^2} - 1\right)\cos^2\theta},$$

$$\text{and } \rho^2 = \frac{b^2}{1 - \left(1 - \frac{b^2}{a^2}\right)\cos^2\phi}.$$

Comparing with the values of u'^2t^2, we see that the luminiferous surface for the extraordinary rays will be a spheroid of revolution, and oblate if μ is greater than μ', but prolate if μ be less than μ'. The former is the case for calc spar, and the latter is the case for ice.

ART. 4. PROP. *To find the retardation of one luminiferous surface behind the other in uniaxal crystals at given points.*

Let $OA = a$, $OB = b$ be the semiaxes of the generating ellipse of the spheroid in fig. (6), which is the luminiferous surface of the extraordinary rays, and taken first as oblate; and $OA' = b$ the radius of the sphere which is the luminiferous surface of the ordinary rays.

Then if the angle ROZ equals i'', we have from the equation to the ellipse,

$$\frac{1}{\rho^2} = \frac{1 + \left(\frac{a^2}{b^2} - 1\right)\cos^2 i''}{a^2}$$

$$= \frac{1}{b^2} - \frac{a^2 - b}{a^2b^2}\sin^2 i'',$$

$$\text{and } \frac{1}{b^2} - \frac{1}{\rho^2} = \frac{\rho^2 - b^2}{b^2\rho^2} = \frac{(\rho - b)(\rho + b)}{b^2\rho^2}$$

$$= \frac{a^2 - b^2}{a^2b^2} \sin^2 i''';$$

$$\therefore \text{ the retardation } \quad \rho - b = \frac{\rho^2(a^2 - b^2)}{a^2(\rho + b)} \sin^2 i'''.$$

Now near the axis Oz, we have $\rho = b$, nearly, and the retardation $pR = \rho - b$

$$= \frac{b(a^2 - b^2)}{2a^2} \sin^2 i''', \qquad \text{nearly.}$$

COR. 1. When the spheroid is prolate, we have

$$\frac{1}{\rho^2} = \frac{1}{a^2} + \frac{a^2 - b^2}{a^2b^2} \sin^2 i''',$$

and the retardation $a - \rho = \frac{a(a^2 - b^2)}{2b^2} \sin^2 i'''$, nearly.

COR. 2. When the light has traversed the plate of the crystal and emerged into air, we have the thickness of the plate equal to b in the first case and equal to a in the second; and also i being the angle of incidence, we have

$$\sin^2 i''' = \frac{\sin^2 i}{\mu^2}$$

near Oz, and the retardation in air is

$$\mu \times pR = \frac{b(a^2 - b^2)}{2\mu a^2} \sin^2 i, \qquad \text{for negative crystals,}$$

$$\mu \times pR = \frac{a(a^2 - b^2)}{2\mu b^2} \sin^2 i, \qquad \text{for positive crystals.}$$

ART. 5. PROP. *To find the relative directions of the two rays in uniaxal crystals near the optic axis, which arise from a single incident ray.*

From the last proposition we have, since $n = n'$, when $i = 0$,

$$n^2v^2 \cos^2 i = u^2 \cos^2 i'$$

$$= u'^2 \cos^2 i''',$$

and $$v^2 \sin^2 i = \mu^4 u^2 \sin^2 i' = \mu'^4 u'^2 \sin^2 i'';$$

∴ dividing, we have $$\mu^4 \tan^2 i' = \mu'^4 \tan^2 i'',$$

or, $$\tan i'' = \frac{\mu^2}{\mu'^2} \tan i', \text{ nearly,} \qquad \text{near the optic axis.}$$

Now in calc spar μ is greater than μ', and therefore i'' is greater than i'. When i is not equal to 0 nearly, the correct values of n and n' must be taken.

When the luminiferous surfaces of the incident rays are planes, those of the refracted ones are so also, and their positions are found by the method of Huygens, as in Part I. Reciprocally the luminiferous surfaces after emergence into air again are planes, perpendicular to the parallel emergent rays.

ART. 6. PROP. *When the surface of a plate of a uniaxal crystal is a principal plane, to determine the direction of the extraordinary ray which corresponds to a given incident ray.*

The refracting surface being a principal plane, let it be taken the plane of xy, and let the optic axis be Ox in the figure, and the crystal a negative one, as calc spar. Then in Art. (3), the luminiferous surface of the extraordinary rays was found to be an oblate spheroid, and in the present case the ray which is incident perpendicularly to the refracting surface will suffer no deviation in the direction by refraction, and the expressions for l', m', n' will be of like form to the last.

Let α'', β'', γ'' be the angles made with Ox, Oy, Oz respectively by the extraordinary ray.

Let α, β, γ be the angles made with Ox, Oy, Oz respectively by the incident ray.

Let ρ be the radius vector of the elliptic section of the oblate spheroid by the plane of xy, and $\varpi = \dfrac{1}{\rho}$;

then putting $$\varpi^2 = \frac{1}{\rho^2} = \frac{\cos^2\theta}{b^2} + \frac{\sin^2\theta}{a^2} = \mu^2\cos^2\theta + \mu'^2\sin^2\theta,$$

where θ is the angle which ρ makes with Ox, the minor axis, and which the plane of refraction makes with the plane of xz,

and $$l'^2 = \frac{1}{\mu^4}, \quad m'^2 = \frac{1}{\mu'^4}, \quad n'^2 = \frac{1}{\mu'^2\varpi^2}\{1 + (\varpi^2 - 1)\sec^2\gamma\},$$
as in Art. (3),

we have $$u'^2\cos^2\alpha'' = l'^2 v^2 \cos^2\alpha$$
$$= \frac{v^2\cos^2\alpha}{\mu^4} \dots\dots\dots\dots\dots\dots\dots\dots \text{(1)},$$
$$u'^2\cos^2\beta'' = m'^2 v^2\cos^2\beta$$
$$= \frac{v^2\cos^2\beta}{\mu'^4} \dots\dots\dots\dots\dots\dots\dots\dots \text{(2)},$$
$$u'^2\cos^2\gamma'' = {}^{\prime 2}\;{}^{2}\cos^2\gamma$$
$$= \frac{v^2}{\mu'^2\varpi^2}(\cos^2\gamma + \varpi^2 - 1) \dots\dots\dots\dots \text{(3)},$$

and $$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$
$$= \frac{u'^2}{v^2}(\mu^4\cos^2\alpha'' + \mu'^4\cos^2\beta'' + \mu'^2\varpi^2\cos^2\gamma'') - \varpi^2 + 1;$$
$$\therefore \frac{v^2}{u'^2} = \frac{\mu^4\cos^2\alpha'' + \mu'^4\cos^2\beta'' + \mu'^2\varpi^2\cos^2\gamma''}{\varpi^2},$$

and this value of $\frac{v^2}{u'^2}$ being substituted in the equations (1), (2) and (3), we have the values of α'', β'', γ'' in terms of α, β and γ.

Let ϕ be the angle which the plane of incidence makes with the plane of *xz*, and in fig. (8) equal to the arc *BC* in the spherical triangle *ABC* upon the surface of a sphere with center *O* and radius unity, where arc $AB = \alpha$, and arc $AC = 90^\circ - \gamma$,

then $$\cos AB = \cos AC \, . \cos BC,$$
or, $$\cos\alpha = \sin\gamma \, . \cos\phi;$$
similarly, $$\cos\beta = \sin\gamma \, . \sin\phi,$$

and for the refracted ray, we have similarly,
$$\cos\alpha'' = \sin\gamma''\cos\theta,$$
$$\cos\beta'' = \sin\gamma''\sin\theta;$$

substituting these values in equations (1) and (2), and dividing (2) by (1), we have

$$\tan\theta = \frac{\mu^2}{\mu'^2}.\tan\phi,$$

which gives the plane of refraction and the value of ϖ^2, by substitution in

$$\varpi^2 = \mu^2\cos^2\theta + \mu'^2\sin^2\theta.$$

Let the plane of incidence be the plane of xz and a principal plane, then $\phi = 0$ and $\theta = 0$,

$$\varpi^2 = \mu^2, \qquad \beta'' = 90^0, \qquad \alpha'' = 90^0 - \gamma'',$$

and

$$\frac{v^2}{u'^2} = \frac{\mu^4\sin^2\gamma'' + \mu'^2\mu^2\cos^2\gamma''}{\mu^2}$$

$$= \mu^2\sin^2\gamma'' + \mu'^2\cos^2\gamma'',$$

and

$$\cos^2\gamma'' = \frac{v^2}{\mu'^2u'^2\varpi^2}(\cos^2\gamma + \varpi^2 - 1)$$

$$= (\mu^2\sin^2\gamma'' + \mu'^2\cos^2\gamma'')\left(\frac{\cos^2\gamma + \mu^2 - 1}{\mu^2.\mu'^2}\right);$$

$$\therefore\ \mu^2\tan^2\gamma'' + \mu'^2 = \frac{\mu^2\mu'^2}{\mu^2 - \sin^2\gamma},$$

and

$$\tan\gamma'' = \frac{\mu'\sin\gamma}{\mu\sqrt{(\mu^2 - \sin^2\gamma)}};$$

when $\gamma = 90^0$, we have γ'' the critical angle, and

$$\tan\gamma'' = \frac{\mu'}{\mu\sqrt{(\mu^2 - 1)}}.$$

In calc spar with Malus's values of μ and μ', we find γ'' in this plane to be 34°. 14′ for the critical angle.

Again, let the plane of incidence be the plane of yz, and the plane of the equator of the spheroid, then $\phi = 90^0$ and $\theta = 90^0$, $\varpi^2 = \mu'^2$; $\alpha'' = 90^0$ and $\beta'' = 90^0 - \gamma''$;

$$\therefore\ \frac{v^2}{u'^2} = \mu'^2(\sin^2\gamma'' + \cos^2\gamma'') = \mu'^2,$$

$$\cos^2\gamma'' = \frac{\cos^2\gamma + \mu'^2 - 1}{\mu'^2} = 1 - \frac{\sin^2\gamma}{\mu'^2},$$

$$\sin^2\gamma'' = \frac{\sin^2\gamma}{\mu'^2},$$

as in ordinary refraction with μ' for refractive index, and putting $\gamma = 90^\circ$, we have the critical angle from

$$\sin\gamma'' = \frac{1}{\mu'};$$

for calc spar in this plane we find the critical angle to be 42°. 23′, whilst that of the ordinary rays in all planes of incidence is 37°. 12′.

If we grind and polish a small plane such as *abc*, fig. 9, a principal plane at a solid angle of a crystal of calc spar, and cover it over with a piece of tin foil having a minute hole in it as marked in the figure, and then form around that minute hole as center, a spherical surface *ABC*, we may observe the appearances shewn in fig. 10. The spherical surface *ABC* being left finely ground but not polished, it serves as a screen to receive the ordinary and extraordinary rays when the small hole in the tin foil is turned towards the sun or flame of a lamp. Opposite the hole we see a single bright spot *a* where the two rays coincide; on each side of it in the principal plane we see that the two rays separate and the ordinary ray, which is polarized in that plane, is the furthest from *a*; whilst in the plane at right angles to it, the ordinary ray is nearest to *a*. These are represented in the figure by the spots marked *e* and *o*. When the piece of crystal is turned about, it is very interesting to see how the ordinary and extraordinary spots turn about each other. The phenomena are in accordance with the above investigations.

ART. 7. Though quartz is ranked with the uniaxal crystals, it has peculiarities which have been discussed in Part I. The two rays are to be considered as elliptically polarized, becoming circularly polarized along the axes of the crystal, and plane polarized at right angles to it. We may consider the sheets of the luminiferous surfaces as splitting each into two, polarized at right angles, as we approach the optic axis, of which one part has a retardation of one-fourth of an interval behind the other; and when the light passes along the axis the two parts of each ray become equal and then constitute circularly polarized light. One of the two rays in this manner becomes circularly left-handed polarized light, and the other circularly right-handed polarized light, and they traverse the axis with different velocities: in right-handed quartz the latter travels quickest, and in left-handed quartz the former travels quickest.

ART. 8. The most general case of a crystal which we can conceive, is that in three rectangular directions the velocities within it shall be all different, and if μ_x is the index for a ray traversing the axis of x and polarized in the plane zx, and μ_x' when it is polarized perpendicular to that plane; μ_y and μ_y' similarly the indices for a ray traversing the axis of y; and also μ_z the index for a ray traversing the axis of z and polarized perpendicular to the plane of xz, μ_z' that when polarized perpendicular to the plane of yz. Then the most general case would be when the quantities μ_x, μ_y, μ_z, μ_x', μ_y', μ_z', were all different. Now we may imagine cases, where the luminiferous surfaces were ellipsoids and one within the other; and other cases where two ellipsoids would intersect. The case of quartz approximates, with peculiarities, to the first of these, but we have no instance of the latter being met with. The case of biaxal crystals occurs as in the next Article.

ART. 9. To consider the compositions of the velocities and polarizations of the luminiferous molecules along the three rectangular axes in biaxal crystals, let OL, in figure 11, represent the velocity of a ray in the axis Ox within the crystal which is polarized in the plane of xz, as represented by the letter ⊢; and in the same way let OL' represent the velocity of the ray in the same direction when polarized in the plane of xy, as represented by the letter ⊣. Let OM and OM' represent in like manner the velocities and polarizations of the rays in Oy; and ON, ON' those of the rays in Oz polarized in the planes of yz and xz respectively, as indicated in the figure.

Now a ray in the quadrant xOy would have its velocity and polarization determined by compounding OL and OM, or OL'

and OM', and the former would be polarized in a plane passing through the ray and the axis Oz, whilst the latter would be polarized in the plane of xy. By the same procedure OM and ON would compound together for the velocity of a ray in the quadrant yOz polarized in that plane; and OM', ON' for the velocity of the ray in the same quadrant when polarized in the plane through the ray and the axis Ox. We have still to consider the composition for rays in the quadrant xOz, and we see that OL must be compounded with ON', and OL' with ON, the former being polarized in the plane xOz, and the latter in a plane passing through the ray and the axis Oy. In this manner the re-composition with the results for the other co-ordinate planes cannot be continuous, when we require the velocity and polarization of a ray passing from O any where in the solid angle contained by the co-ordinate planes.

We see hence how the break in the trace of the luminiferous surface in the plane of xz, arises in biaxal crystals.

ART. 10. PROP. *To determine which of the two rays belongs to the inner and which to the outer sheet, of the luminiferous surfaces, in a plate of arragonite, cut perpendicular to the line bisecting the optic axes.*

Referring to the description of the experiments and figures (25) and (26), Part I.,

Let μ_x, μ_y, μ_z be the greater indices in the axes Ox, Oy, Oz respectively in figure (11).

Let μ_x', μ_y', μ_z' be the indices in the axes Ox, Oy, Oz respectively for rays polarized perpendicularly to the former, and so that v being the velocity of the incident rays, we have for the inner sheet $OL =$ space in Ox, $OM =$ space in Oy, $ON =$ space in Oz, described in the same time; and for the outer sheet, $OL' =$ space in Ox, $OM' =$ space in Oy, $ON' =$ space in Oz, described in the same time.

In order to find which rays belong to the inner and which to the outer sheet, we must employ the method of Art. (5) for uniaxal crystals with the sphere and spheroid for luminiferous surfaces; and assume imaginary ordinary rays for the purpose of comparison. In that article we obtained the expression,

$$\frac{\tan i''}{\tan i'} = \frac{\mu^2}{\mu'^2}.$$

Now if i' and i'' are the angles in the plane of xz made with the axis Oz by the two refracted rays in biaxal crystals, let i_1' and i_1'' be the angles made by the supposed ordinary rays and refractive indices μ_z', μ_z respectively. Since OL compounds with ON' in fig. (11), and OL' with ON. Also i' being the angle for rays polarized in the plane of incidence, and i'' the angle for those polarized perpendicularly to it.

Then $$\frac{\tan i'}{\tan i_1'} = \frac{\mu_z'^2}{\mu_x^2} \quad \text{and} \quad \frac{\tan i''}{\tan i_1''} = \frac{\mu_z^2}{\mu_x'^2}.$$

Dividing one of these expressions by the other, we have

$$\frac{\tan i''}{\tan i'} \times \frac{\tan i_1'}{\tan i_1''} = \frac{\mu_z^2 . \mu_x^2}{\mu_z'^2 . \mu_x'^2}.$$

But from the property of ordinary refraction, and i equal the angle of incidence,

$$\frac{\sin i_1'}{\sin i_1''} = \frac{\dfrac{\sin i}{\mu_z'}}{\dfrac{\sin i}{\mu_z}} = \frac{\mu_z}{\mu_z'}$$

$$= \frac{\tan i_1'}{\tan i_1''} \text{ nearly,} \qquad \text{when } i_1' \text{ and } i_1'' \text{ are small.}$$

Therefore near Oz, by substituting, we have

$$\frac{\tan i''}{\tan i'} = \frac{\mu_z}{\mu_z'} . \frac{\mu_x^2}{\mu_x'^2}.$$

Referring to figure (11), the rays for the outer sheet near Oz have μ_x and μ_z' for indices, and i' for angle of refraction; whilst the rays for the inner sheet have μ_x' and μ_z for refractive indices; and i'' for the angle of refraction, and we have

$$\frac{\mu_z}{\mu_z'} > 1, \quad \frac{\mu_x^2}{\mu_x'^2} > 1.$$

Therefore i''' is greater than i'', in the plane of xz, and near Oz; and the rays polarized in the plane of xz are nearer to the axis, and belong to the outer sheet, whilst the rays polarized perpendicular to the plane of xz are further from the axis and belong to the inner sheet. From this discussion we see on comparing with figures (25) and (40), Part I, that the eyelet-hole is in the outer sheet, and the conical point in the inner one, as shewn in figure (26), the plane of polarization of the rays turning through ninety degrees as we pass from one side of the eyelet-hole or conical point, to the opposite side; from the nature of the composition which has been described, and according with M. Biot's law of the polarizations.

Art. 11. Prop. *To find the equations of the luminiferous surfaces in biaxal crystals when the refracting surface is perpendicular to the line bisecting the angle between the optic axes.*

As in the last two articles, let μ_x, μ_y, μ_z be the largest indices in the axes Ox, Oy, Oz respectively; and μ_x', μ_y', μ_z' the smallest indices in the axes respectively.

Let α, β, γ, be the angles which an incident ray makes with Ox, Oy, Oz, respectively; α', β', γ'; α'', β'', γ'' those which the two refracted rays make with them.

Let v be the velocity of the luminiferous molecules in the incident ray SO, figure (12).

Let u be the velocity in the refracted ray OR' belonging to the outer sheet, and making angles α', β', γ' with the axes.

Let u' be that in the refracted ray OR belonging to the inner sheet, and making angles α'', β'', γ'', with the axes respectively.

Then l, l', m', n', being the coefficients which multiplied into the velocities in the axes respectively of the incident ray, give the velocities of the refracted rays of the outer sheet; and θ being the angle which the plane passing through Oz and OR of the inner sheet makes with the plane of xz; θ' that which the plane passing through Oz and OR' of the outer sheet makes with the plane of xz;

$$\text{let} \qquad \varpi^2 = \mu_x^2 \cos^2\theta + \mu_y^2 \sin^2\theta,$$

and $\varpi'^2 = \mu_x'^2 \cos^2\theta' + \mu_y'^2 \sin^2\theta'$, we have

$$l^2 = \frac{1}{\mu_x^4}, \quad \text{and } n'^2 = \frac{1 + (\varpi'^2 - 1)\sec^2\gamma}{\mu_z'^2 . \varpi'^2} \text{ within the optic axes,}$$

$$l'^2 = \frac{1}{\mu_x'^4}, \quad \text{and } n'^2 = \frac{1 + (\varpi'^2 - 1)\sec^2\gamma}{\mu_z^2 . \varpi'^2} \text{ without the optic axes,}$$

$$\text{also} \qquad m'^2 = \frac{1}{\mu_y'^4}.$$

Let $\bar{\gamma}$ be the value of γ'' in the plane of xz at the optic axis, or the angle AOz in the figure. Then in the traces of the luminiferous surface on the plane of xz, there are breaks at A and A' the optic axes, and expressing the discontinuity by Mr Rawson's method*, let χ and χ' be the operating functions of discontinuity, and

$$\chi = \frac{1}{2}\left(1 + \frac{\bar{\gamma} - \gamma''}{\bar{\gamma} \sim \gamma''}\right),$$

$$\chi' = \frac{1}{2}\left(1 - \frac{\bar{\gamma} - \gamma''}{\bar{\gamma} \sim \gamma''}\right).$$

Now $\bar{\gamma} \sim \gamma''$ being always positive, we have $\chi = 1$ for all values of γ'' from Oz to the optic axis AO, $\chi = 0$ for all values of γ'' from AO to Ox. In the same way $\chi' = 0$ for all values of γ'' less than $\bar{\gamma}$, and $\chi' = 1$ for all values of γ'' greater than $\bar{\gamma}$.

As before, we have

$$u^2 \cos^2\alpha' = l^2 v^2 \cos^2\alpha; \quad \text{or } \cos^2\alpha = \frac{\mu_x^4 u^2 \cos^2\alpha'}{v^2},$$

$$u^2 \cos^2\gamma' = n'^2 v^2 \cos^2\gamma; \quad \text{or } \cos^2\gamma = \frac{\mu_z'^2 \varpi'^2 u^2 \cos^2\gamma'}{v^2} - \varpi'^2 + 1,$$

within the optic axes; and

$$u^2 \cos^2\alpha' = l'^2 v^2 \cos^2\alpha; \quad \text{or } \cos^2\alpha = \frac{\mu_x'^4 u^2 \cos^2\alpha'}{v^2},$$

$$u^2 \cos^2\gamma' = n'^2 v^2 \cos^2\gamma; \quad \text{or } \cos^2\gamma = \frac{\mu_z^2 \varpi'^2 u^2 \cos^2\gamma'}{v^2} - \varpi'^2 + 1,$$

without the optic axes.

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* See the *Memoirs of the Literary and Philosophical Society of Manchester*, New Series, Vol. VIII. for 1847.

Also $u^2 \cos^2\beta' = m'^2 v^2 \cos^2\beta$; or $\cos\beta = \dfrac{\mu_y'^4 u^2 \cos^2\beta'}{v^2}$.

Therefore substituting we have

$$\begin{aligned} 1 &= \cos^2\alpha + \cos^2\beta + \cos^2\gamma \\ &= \left\{\frac{u^2}{v^2}(\mu_x{}^4 \cos^2\alpha' + \mu_z'^2 \varpi'^2 \cos^2\gamma') - \varpi'^2 + 1\right\}\chi \\ &+ \left\{\frac{u^2}{v^2}(\mu_x'^4 \cos^2\alpha' + \mu_z{}^2 \varpi'^2 \cos^2\gamma') - \varpi'^2 + 1\right\}\chi' + \mu_y'^4 \cos^2\beta' \frac{u^2}{v^2}. \end{aligned}$$

Now striking out 1 on both sides, whether χ or χ' is effective, and dividing down u^2, we have

$$\begin{aligned} \frac{\varpi'^2}{u^2} = \frac{1}{v^2}\{&(\mu_x{}^4 \cos^2\alpha' + \mu_z'^2 \varpi'^2 \cos^2\gamma')\chi \\ &+ (\mu_x'^4 \cos^2\alpha' + \mu_z{}^2 \varpi'^2 \cos^2\gamma')\chi' + \mu_y'^4 \cos^2\beta'\}, \end{aligned}$$

which, multiplying u^2 and v^2 each by the time squared (t^2), is the equation of the *outer* sheet of the luminiferous surface.

The same procedure gives the equation of the *inner* sheet as follows:

$$\begin{aligned} \frac{\varpi^2}{u'^2} = \frac{1}{v^2}\{&(\mu_x'^4 \cos^2\alpha'' + \mu_z{}^2 \varpi^2 \cos^2\gamma'')\chi \\ &+ (\mu_x{}^4 \cos^2\alpha'' + \mu_z'^2 \varpi^2 \cos^2\gamma'')\chi' + \mu_y{}^4 \cos^2\beta''\}. \end{aligned}$$

These are represented in figure 12.

ART. 12. PROP. *To find the angle between the optic axes in biaxal crystals.*

At the conical point of the inner sheet the values of u' are equal for the two parts of the expression affected by χ and χ'. Then we have at the conical point

$$\beta'' = 90^0,\ \alpha'' = 90^0 - \bar{\gamma},\ \gamma'' = \bar{\gamma};$$

also in the plane of xz, we have $\theta = 0$ and $\varpi^2 = \mu_x{}^2$; then equating the two parts of the value of $\dfrac{\varpi^2}{u'^2}$ we find

$$\mu_x'^4 \tan^2\bar{\gamma} + \mu_z{}^2 \mu_x{}^2 = \mu_x{}^4 \tan^2\bar{\gamma} + \mu_z'^2 \mu_x{}^2;$$

$$\therefore\ \tan^2\bar{\gamma} = \frac{\mu_x^2\,(\mu_z^2 - \mu_z^{\prime 2})}{\mu_x^4 - \mu_x^{\prime 4}},$$

which gives the value of $\bar{\gamma}$, and when $\mu_z^2 = \mu_z^{\prime 2}$ the crystal becomes uniaxal and $\bar{\gamma} = 0$.

Mica is generally a biaxal crystal with large separation of the axes, but certain kinds have the axes very near together, and some are to all appearance uniaxal. In some crystals, such as topaz, the velocities of the polarized rays are the reverse of those taken as in fig. 11 to represent the case of arragonite, and the equation must be modified accordingly. Arragonite is sometimes called a negative and topaz a positive crystal.

ART. 13. PROP. *To find the angular separation of the two rays within the optic axes, which arise from the same incident ray, and to find the radius of the eyelet-hole in the outer sheet.*

If ϕ be the angle between the two rays, we have

$$\cos\phi = \cos\alpha' \cos\alpha'' + \cos\beta' \cos\beta'' + \cos\gamma' \cos\gamma'',$$

and from Art. 11, within the optic axes,

$$\cos^2\alpha = \mu_x^4 \frac{u^2}{v^2} \cos^2\alpha' = \mu_x^{\prime 4} \frac{u^{\prime 2}}{v^2} \cos^2\alpha'',$$

$$\cos^2\beta = \mu_y^{\prime 4} \frac{u^2}{v^2} \cos^2\beta' = \mu_y^4 \frac{u^{\prime 2}}{v^2} \cos^2\beta'',$$

$$\cos^2\gamma = \mu_z^{\prime 2} \varpi^{\prime 2} \frac{u^2}{v^2} \cos^2\gamma' - \varpi^{\prime 2} + 1 = \mu_z^2 \varpi^2 \frac{u^{\prime 2}}{v^2} \cos^2\gamma'' - \varpi^2 + 1\,;$$

substituting the values of $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ from these expressions, we have

$$\cos\phi = \frac{u'}{u}\left[\frac{\mu_x^{\prime 2}}{\mu_x^2} \cos^2\alpha'' + \frac{\mu_y^2}{\mu_y^{\prime 2}} \cos^2\beta'' \right.$$
$$\left. + \frac{\mu_z \varpi}{\mu_z' \varpi'} \cos^2\gamma'' \sqrt{\left\{1 - \frac{(\varpi^2 - \varpi^{\prime 2})\, v^2}{\mu_z^2 \varpi^2 u^{\prime 2} \cos^2\gamma''}\right\}}\,\right],$$

and a similar expression may be obtained in terms of α', β', γ'.

Let $\bar{\phi}$ be the separation at the optic axis which is the radius of the eyelet-hole, when $\beta'' = 90^\circ$ nearly, and $\alpha'' = 90^\circ - \bar{\gamma}$; then omitting the small term with the factor $(\varpi^2 - \varpi'^2)$ under the sign of the square root, we have, since then $u' = u$, $\varpi = \mu_z$, $\varpi' = \mu_z'$,

$$\cos\bar{\phi} = \frac{\mu_z'^2}{\mu_z^2}\sin^2\bar{\gamma} + \frac{\mu_z\mu_x}{\mu_z'\mu_x'}\cos^2\bar{\gamma},$$

which gives $\bar{\phi}$ when $\bar{\gamma}$ is known.

ART. 14. PROP. *To find the retardation of one sheet behind the other when the rays have coincident directions.*

Taking $\chi' = 0$, and $\chi = 1$ in the equations of Art. 11, we have

$$\frac{\varpi'^2}{u^2} = \frac{1}{v^2}\{\mu_x^4\cos^2\alpha' + \mu_z'^2\varpi'^2\cos^2\gamma' + \mu_y'^4\cos^2\beta'\},$$

$$\frac{\varpi^2}{u'^2} = \frac{1}{v^2}\{\mu_x'^4\cos^2\alpha'' + \mu_z^2\varpi^2\cos^2\gamma'' + \mu_y^4\cos^2\beta''\},$$

and putting for coincidence of directions $\alpha'' = \alpha'$, $\beta'' = \beta'$, $\gamma'' = \gamma'$, we have, taking $\varpi' = \varpi$ nearly,

$$\frac{1}{u'^2} - \frac{1}{u^2} = \frac{1}{\varpi^2 v^2}\{(\mu_x'^4 - \mu_x^4)\cos^2\alpha' + \varpi^2(\mu_z^2 - \mu_z'^2)\cos^2\gamma' \\ + (\mu_y^4 - \mu_y'^4)\cos^2\beta'\},$$

but, $$\tan^2\bar{\gamma} = \frac{\varpi^2(\mu_z^2 - \mu_z'^2)}{\mu_x^4 - \mu_x'^4};$$

therefore, $$\sin^2\bar{\gamma} = \frac{\varpi^2(\mu_z^2 - \mu_z'^2)}{\mu_x^4 - \mu_x'^4 + \varpi^2(\mu_z^2 - \mu_z'^2)},$$

$$\cos^2\bar{\gamma} = \frac{\mu_x^4 - \mu_x'^4}{\mu_x^4 - \mu_x'^4 + \varpi^2(\mu_z^2 - \mu_z'^2)};$$

putting $$R = \frac{\mu_x^4 - \mu_x'^4 + \varpi^2(\mu_z^2 - \mu_z'^2)}{\varpi^2},$$

we have

$$\frac{1}{u'^2} - \frac{1}{u^2} = \frac{R}{v^2}\{-\cos^2\bar{\gamma}\cos^2\alpha' + \sin^2\bar{\gamma}\cos^2\gamma' + \frac{1}{R\varpi^2}(\mu_y^4 - \mu_y'^4)\cos^2\beta'\},$$

putting $Q = \dfrac{\mu_y^4 - \mu_y'^4}{R\varpi^2}$, and since $\alpha' = 90^\circ - \gamma' +$ a small quantity near the plane of xz, where also $\cos\beta'$ is small, let

$$\cos^2\alpha' = \sin^2\gamma' - \Delta,$$

where Δ is small, then the above expression becomes

$$\frac{1}{u'^2} - \frac{1}{u^2} = \frac{R}{v^2}\{-\cos^2\bar{\gamma}\,(\sin^2\gamma' - \Delta) + \sin^2\bar{\gamma}\cos^2\gamma' + Q\cos^2\beta'\}$$

$$= \frac{R}{v^2}\{(\cos\gamma'\sin\bar{\gamma} + \cos\bar{\gamma}\sin\gamma')\,(\cos\gamma'\sin\bar{\gamma} - \cos\bar{\gamma}\sin\gamma')$$
$$+ Q\cos^2\beta' + \cos^2\bar{\gamma}\,.\,\Delta\}$$

$$= \frac{R}{v^2}\{\sin(\bar{\gamma} + \gamma')\,.\,\sin(\bar{\gamma} - \gamma') + Q\cos^2\beta' + \Delta\,.\,\cos^2\bar{\gamma}\}.$$

In figure 13, let A, B, A', P, be points on the surface of a sphere, whose center is O and radius unity, also A, A' being the optic axes; then OP being a refracted ray, let $PB = \gamma'$, $AP = \omega$, $A'P = \omega'$, and near the plane of xz we have

$$AP = AB - BP \text{ nearly, or } \omega = \bar{\gamma} - \gamma',$$

$$A'P = A'B + BP \text{ nearly, or } \omega' = \bar{\gamma} + \gamma',$$

and $$\frac{1}{u'^2} - \frac{1}{u^2} = \frac{R}{v^2}\sin\omega'\,.\,\sin\omega \text{ nearly,}$$

whence $$u - u' = \frac{u'^2\,.\,u^2}{u' + u}\left(\frac{1}{u'^2} - \frac{1}{u^2}\right)$$

$$= \frac{u^3 R}{2v^2}\sin\omega\,.\,\sin\omega' \text{ nearly;}$$

multiplying each side by the time in traversing OP we have the retardation δ as follows, if T be the thickness of crystal traversed by the ray, and

$$T = u \times \text{time},$$

$$\frac{u^2}{v^2} = \frac{1}{\mu_s^2};$$

$$\therefore\ \delta = \frac{T\,.\,R}{2\mu_s^2}\sin\omega'\,.\,\sin\omega,$$

which is M. Biot's approximate law for the retardation in biaxal crystals, and may be put under the form

$$\delta = k \,.\, AP \times A'P.$$

If we put $\chi' = 1$ and $\chi = 0$, we have for points without the optic axes the same expression for the value of δ, which is therefore general for points near the plane of the optic axes.

Where the rays have emerged into air again we shall have the retardation equal to that immediately before emergence multiplied by the refractive index, and this may be supposed to be contained in the coefficient k.

In order to compare the expressions which have been found for $\bar{\gamma}$ and ϕ in Arts. 12 and 13, with experiments, we require to know the values of the six refractive indices, or if any of them are equal it must be determined experimentally. Whilst it was supposed that all double refracting crystals were of the same class as calc spar, or were uniaxal, M. Malus* concluded that the single axis in arragonite was parallel to the axis of the prism of the crystal, and he measured the greatest and least values of the refractive index to be 1·693135 and 1·534802. In sulphate of barytes he concluded the axis to be parallel to the small diagonal of the base of the crystal, and the greatest and least values of the refractive index to be 1·646842 and 1·635242.

M. Rudberg†, considering that three of the refractive indices according to M. Fresnel's theory were sufficient to determine the elasticity of the hypothetical æther in the principal axes within biaxal crystals, proceeded to determine them for arragonite and topaz, with sets of prisms cut from the same crystals, and having their refracting edges parallel to these axes. These suppositions required only three prisms to be made of each substance, and with them the refractive indices were found for the different fixed lines of the solar spectrum. Those for the fixed line F

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* *Mémoires des Savans Etrangers,* Tom. II. for 1811.

† *Phil. Mag.* for August, 1832, page 140.

coincide the nearest with the values of M. Malus above. For the fixed line *E* the values are as in the table.

Arragonite.	Topaz.
1·53264	1·62408
1·69084	1·61452
1·68634	1·61668

If these crystals were conformable to the supposed rule of the undulatory theory, it still would require to be examined for others belonging to the oblique prismatic system of crystallization.

Now the above values give the angle between the optic axes within the crystal or twice $\bar{\gamma}$, from M. Fresnel's formula, to be 19°. 53′ and in air 33°. 51′. The measures in air were found by M. Rudberg to be about 32° and by Professor Lloyd to be 31°. 56′*. Sir D. Brewster† first measured the angle between the axes within a crystal to be 18°. 18′, but on a careful re-examination found it 17°. 33′, and in air 29°. 56′. These results differ so considerably from the theoretical values that we may conclude the six values of μ will not be found eventually to be reducible to three in arragonite.

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[*] *Phil. Mag.* for February, 1833, page 112.

[†] *Ibid.* for August, 1832, page 147.

CHAPTER II.

ON THE POLARIZATION OF LIGHT AT TRANSPARENT SURFACES.

ON the corpuscular theory, light must be considered to consist of molecules emitted in sheets following each other at the distance λ, the sheets being perpendicular to the rays in ordinary media; and we require to consider each molecule as compounded of atoms of various subtile matters, in order to account for the different properties of the rays as chemical, luminiferous, and heat-making. Such molecules, like the molecules of crystallized bodies, must also have axial and equatorial directions, and may have three principal axes in some cases. We find the phenomena of polarization to indicate that *the axes* of uniaxal molecules are *in the luminiferous surfaces*, and the planes of the equators are perpendicular to them and pass through the rays of light in ordinary media.

When a beam of light is polarized in any plane, all the equators of the molecules, and therefore all the axes, are parallel. The plane of the polarization is the plane passing through *a ray* and *the equator* of the luminiferous molecules of that ray. The plane passing through the ray and the axis of the molecule is consequently at right angles to the plane of polarization.

When a molecule arrives at O, figure (14), let qSe be the plane of its equator; then the plane of polarization of the ray SO is the plane qSe.

Let Op be the direction of the axis of the molecule; NOS any other plane making the angle $NSq = a$ with the plane of polarization; Sp an arc of a great circle through p. Then the angle $NSp = 90^\circ - a$.

ART. 15. PROP. *To investigate the law of the polarization of a beam of light after reflexion at a transparent surface.*

Let OS be the incident ray in figure (15), OT the reflected ray, OR the refracted ray, and NON' the normal to the refracting surface at O; the axes of the molecules must be *in* the luminiferous surfaces for these rays.

Draw a sphere with radius unity and center O, meeting OS in S, OT in T, and OR produced backwards in d. Let Op be the direction of the axis of the incident molecule, and Ot that of the reflected molecule; then the angles SOp and TOt are each 90°, from the previous discussion. Now from experiment it appears that the axis of the molecule in passing from the direction Op to Ot goes through a position Oa in the luminiferous surface of the refracted ray ROd; and that OS, Op, and Oa are in one plane, whilst Oa, Ot, and OT are in another.

Draw the arcs of great circles SNT, Spa, da, Tta with center O and radius unity; we have Sp, da, Tt, each 90°. Let the angle of incidence $SON = i =$ the angle of reflexion TON; and the angle of refraction $RON' = i' =$ angle NOd.

Let α be the angle which the plane of polarization of the incident ray makes with the plane of incidence, β the angle which the plane of polarization of the reflected ray makes with it; then, as before explained, the angle $NSp = 90° - \alpha$, and the angle $NTt = 90° - \beta$.

Now in the spherical triangles Sad, Tad, since $da = 90°$, we have

$$\cos NSa = -\cot Sd \,.\, \cot Sa,$$

$$\text{or } \sin\alpha = -\cot(i - i')\cot Sa, \ldots\ldots\ldots\ldots\ldots(1)$$

$$\cos NTa = -\cot Td \,.\, \cot Ta,$$

or

$$\sin\beta = -\cot(i + i') \,.\, \cot Ta, \ldots\ldots\ldots\ldots\ldots(2)$$

also

$$\sin Sa = \frac{\sin STa}{\sin TSa}\sin Ta$$

$$= \frac{\cos\beta}{\cos\alpha}\sin Ta. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Eliminating *Sa* and *Ta* from the equations (1), (2), and (3), we have

$$\frac{\tan^2\beta}{\cos^2(i+i')} = \frac{\tan^2\alpha}{\cos^2(i-i')},$$

or extracting the square root and retaining the negative sign, for β considered positive in the same direction as α,

$$\tan\beta = -\tan\alpha\,.\,\frac{\cos(i+i')}{\cos(i-i')},$$

which is Fresnel's formula.

When $(i+i') = 90^\circ$, $\cos(i+i') = 0$, then $\tan\beta = 0$ whatever $\tan\alpha$ may be; that is, the reflected light is polarized in the plane of incidence, even if the incident beam contained rays polarized in different planes. We see that when $i+i' = 90^\circ$, the reflected and refracted rays are at right angles, and

$$i = 90^\circ - i', \quad \therefore \cos i = \sin i' = \frac{\sin i}{\mu},$$

$$\therefore \tan i = \mu,$$

which is Brewster's law for the polarizing angle at transparent surfaces.

Fresnel's law as above has been examined by M. Arago and Sir D. Brewster[*], and maintained by them to be strictly accurate; M. Jamin however, by more accurate experiments, has found it to be only a first approximation[†] for the greater number of transparent bodies, but is the accurate law when the refractive index is 1·46.

Art. 16. Prop. *To investigate the law of the polarization of a beam of light after refraction at the surface of a transparent body.*

Let the letters in figure 16 refer to the same points as in figure 15.

Then *Oa* being the direction of the axis of the refracted molecule, and $da = 90^\circ$, the angle adS equals 90° minus the angle which the plane of polarization of the refracted ray makes with the plane of incidence; let γ be this angle, or $\gamma = 90^0 - \angle\, adS$,

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[*] *Phil. Trans.* for 1830, page 137.

[†] *Annales de Chimie et de Physique* for 1850, Tome XXIX. page 263.

$$\text{then} \qquad \frac{\sin Sa}{\sin adS} = \frac{\sin ad}{\sin aSd}, \text{ or } \sin Sa = \frac{\cos\gamma}{\cos\alpha};$$

$$\text{again,} \qquad \cos aSd = -\cot Sd \,.\, \cot Sa,$$

$$\text{or} \qquad \tan Sa = -\frac{\cot(i-i')}{\sin\alpha}.$$

Eliminating Sa between these equations, we have

$$\frac{\cos^2\gamma}{\cos^2\alpha} = \frac{\cot^2(i-i')}{\sin^2\alpha + \cot^2(i-i')},$$

$$\text{or} \qquad \cot^2\alpha\,(1+\tan^2\gamma) = \frac{1}{\cot^2(i-i')} + \frac{1}{\sin^2\alpha};$$

$$\text{whence} \qquad \cot^2\alpha = \frac{\cot^2\gamma}{\cos^2(i-i')},$$

$$\text{and} \qquad \cot\gamma = \pm\cot\alpha \,.\, \cos(i-i'),$$

$$\text{or} \qquad \tan\gamma = \pm\frac{\tan\alpha}{\cos(i-i')},$$

which gives γ as required, the lower sign to be taken when γ is measured in the like direction to α.

ART. 17. PROP. *To investigate the law of the polarization of a beam of light after emergence from a plate of a refracting medium.*

In figure 17, let $SOO'R$ be the ray which has traversed the refracting plate, being incident at O and emergent at O'. Let $O'b$ be the direction of the axis of the luminiferous molecule at emergence. Let $O'R$ be produced backwards to A, and let a sphere be drawn with radius unity and center O', meeting the ray OO' within the medium at d, then the points d, A, and b, being upon the surface of the sphere with radius unity and center O', complete the spherical triangle Abd. We have, since $O'b$ is in the luminiferous surface of the emergent ray, $Ab = 90^0$.

If δ is the angle which the plane of polarization of the emergent ray makes with the plane of incidence, we have the angle $dAb = 90^\circ - \delta$; also as in the last proposition the $\angle\, bdA = 90^\circ - \gamma$.

To find δ we have

$$\frac{\sin Adb}{\sin Ab} = \frac{\sin dAb}{\sin db}, \text{ or } \cos\gamma = \frac{\cos\delta}{\sin db},$$

$$\therefore \frac{1}{\sin^2 db} = 1 + \cot^2 db = \frac{\cos^2\gamma}{\cos^2\delta}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

again, $\cos Adb = -\cot Ad \,.\, \cot db,$

or $\sin\gamma = -\cot(i - i') \,.\, \cot db, \ldots\ldots\ldots\ldots\ldots\ldots(2)$

and eliminating db between (1) and (2) we have

$$\tan^2\delta = \frac{\tan^2\gamma}{\cos^2(i - i')},$$

and from the last proposition $\tan\gamma = \pm \dfrac{\tan\alpha}{\cos(i - i')}$,

$$\therefore \tan\delta = \pm \frac{\tan\gamma}{\cos(i - i')} = \frac{\tan\alpha}{\cos^2(i - i')},$$

and the angle which the plane of polarization of the emergent ray makes with the plane of incidence increases as $(i - i')$ increases.

If a beam of ordinary light is incident upon a plate, the emergent beam is never completely polarized in one plane; but at high incidences, when $\cos^2(i - i')$ is small, the polarization approximates towards being perpendicular to the plane of incidence.

If $\alpha = 0$, then $\delta = 0$; and if $\alpha = 90^\circ$, then $\delta = 90^\circ$ also.

ART. 18. PROP. *To investigate the law of the polarization of a beam of light internally reflected within a refracting plate.*

Let $O'B$ be, in figure 18, a ray reflected at O' within the plate, and $O'b$ as in the last proposition.

A sphere being drawn with center O' and radius unity, let it meet the reflected ray $O'B$ in c, and the emergent ray produced backwards in A. Complete the spherical triangle Abc. Then if ϵ be the angle which the plane of polarization of $O'B$ makes with the plane of incidence, we have the angle $Acb = 90^\circ - \epsilon$, since the axis of the luminiferous molecule will be 90° from the point c, and in the plane $O'bc$, as in the other cases.

Now the arc $Ac = i + i'$, and

$$\frac{\sin cb}{\sin bAc} = \frac{\sin Ab}{\sin bcA};$$

$$\therefore \ \frac{1}{\sin^2 cb} = 1 + \cot^2 cb = \frac{\cos^2 \epsilon}{\cos^2 \delta} \ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again, $\cos bcA = -\cot Ac \,.\, \cot cb$,

$$\text{or } \cot bc = -\frac{\sin \epsilon}{\cot (i + i')}; \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

eliminating cb, we have

$$\tan^2 \epsilon = \tan^2 \delta \,.\, \cos^2 (i + i'),$$

extracting the square root and substituting the value of δ, we have

$$\tan \epsilon = \pm \tan \delta \,.\, \cos (i + i')$$

$$= \pm \frac{\tan \alpha}{\cos (i - i')} \cdot \frac{\cos (i + i')}{\cos (i - i')}.$$

ART. 19. PROP. *To investigate the law of the polarization of the emergent ray BT' of fig.* 16, *which has undergone internal reflexion.*

Referring to the figure of the last article, where ϵ is the angle made by the plane of polarization with the plane of incidence, if we look from B towards O', we shall have $-\epsilon$ for ϵ, and applying this value of ϵ in place of γ in the expression for the emergent ray RO', fig. 17, where we found

$$\tan \delta = \pm \frac{\tan \gamma}{\cos (i - i')}.$$

Putting ζ for the angle which the plane of polarization of BT' makes with the plane of incidence, we have

$$\tan\zeta = \mp\frac{\tan\epsilon}{\cos(i-i'')}$$

$$= \pm\frac{\tan\alpha}{\cos^2(i-i'')}\cdot\frac{\cos(i+i'')}{\cos(i-i'')}.$$

Now when $(i+i'') = 90^0$ for the polarizing angle of the first reflected ray at O, we have $\tan\ \zeta = 0$; or the ray emergent at B is also polarized in the plane of incidence.

As $(i-i'')$ becomes larger, $\cos(i-i'')$ is smaller, and $\tan\zeta$ becomes larger, and when $\frac{\cos(i+i'')}{\cos^3(i-i'')}$ equals unity, we have $\tan\zeta = \pm\tan\alpha$, and $\zeta = \pm\alpha$, or the angle for the incident ray is also that for the emergent ray.

When $\cos(i-i'')$ is still smaller, then ζ is greater than $\pm\alpha$, but the quantity of light reflected internally becomes small on account of the large quantity reflected at the first surface.

In the same way, by successive operations, the plane of polarization of a ray emergent at either the first or second surface may be determined after any number of internal reflexions.

In these investigations the change of the plane of polarization has only been considered. The depth within the refracting surface at which reflexion takes place may be evidently different for a ray polarized in the plane of incidence to what it is for a ray polarized perpendicularly to it, and hence a reflected beam may exhibit, as M. Jamin* has found, elliptically polarized light, when the surface analyzes the incident beam into two before reflexion, which are polarized at right angles to each other.

ART. 20. PROP. *To investigate the polarization of the light which has undergone total reflexion within a transparent medium.*

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* *Annales de Chimie et de Physique* for 1850, Tome XXIX. page 263.

When total reflexion takes place within a transparent medium we should expect the totally reflected light to consist of two parts, which represent the reflected and refracted rays of Arts. 17 and 18. But the expression for the emergent ray was

$$\tan\delta = \pm\frac{\tan\gamma}{\cos(i-i')},$$

where δ is the angle made by the plane of polarization of the emergent ray with the plane of incidence, and γ is that made by the plane of polarization of the incident ray within the medium.

Since the phenomena of reflexion generally are deducible from those for refraction by putting $\mu=-1$, and therefore $i'=-i$, we have

$$\tan\delta = \pm\frac{\tan\gamma}{\cos 2i'},$$

which may be also easily obtained by a direct solution, drawing Oa the axis in the plane AOa and joining Ba in fig. 19. Now when the angle of incidence i' within the medium is 45°, we have

$$\tan\delta = \infty,$$

whatever value γ may have otherwise than zero; or the representative of the refracted ray is polarized perpendicular to the plane of incidence.

Again, for the portion of the totally reflected light, which is the representative of the internally reflected rays generally, the refracted ray ceasing as a separate ray, the axes of the luminiferous molecules must pass at once from being perpendicular to the incident ray to being perpendicular to the reflected ray; or if Op in fig. 19, represents the axis of the molecule in the incident ray AO; Ot that in the reflected ray OB; then completing the spherical triangle ApB we have Ap and Bt each equal to 90°, and Ot is in the plane BOp.

Let the angle $pBA = 90^\circ - \epsilon$, and angle $BAp = 90^\circ - \gamma$ as before, we have

$$\sin\epsilon = -\cot 2i' \cot Bp,$$

$$\sin\gamma = \frac{\cos Bp}{\sin 2i'};$$

therefore eliminating *Bp*, we have

$$\sin \epsilon = -\sin \gamma . \frac{\cos 2i''}{\sqrt{(1 - \sin^2 \gamma \sin^2 2i'')}},$$

and when $i'' = 45°$, $\sin \epsilon = 0$, and $\epsilon = 0$; or this portion of the reflected light is polarized in the plane of incidence and reflexion.

We find this latter portion of the totally reflected light to be retarded a space $\frac{\lambda}{8}$ behind the former.

It appears from this investigation that the rhombs to procure circular polarized light should have their acute angles 45°, and hence the results stated at page 38, Part I.

The constitution of circularly and elliptically polarized light belongs to the subject of interference, and will be found discussed under that head.

CHAPTER III.

ON THE INTENSITY OF THE LIGHT REFLECTED AT POLISHED SURFACES.

THE complete failure of the formula $\left(\frac{\mu-1}{\mu+1}\right)^2$ which was found y Dr Young, M. Poisson, and M. Fresnel, for the intensity of the eflexion from a polished surface at a perpendicular incidence, ccording to the undulatory theory, was pointed out in Part I., age 116. An easy method by which this failure may be tested y a comparative photometer was also shewn.

ART. 21. The results of experiments by photometry, made y the author, are given in the same chapter for the reflected ight at the first surface of crown, plate, and flint glass, in the able; these results are the ordinates of an hyperbola mPn, as in ig. 20.

Let O be the point where a ray of light SpO is incident on he reflecting surface xOx'; Oy being at right angles to Ox, we ave the angle of incidence i equal to SOy.

Take a distance $OA = 100$ parts, and let 100 rays be sup-osed incident in a beam near O.

Describe a circular arc ApB, and from p draw the line pPM erpendicular to Ox. Let P be such a point that having $A = r$, $PM = y$, $OM = x$, they are the ordinates of the hyper-ola whose equation is

$$y = a + \frac{c^2}{r + b - x},$$

then giving the constants a, b, c the values which belong to the particular refracting substance, we have $PM = y$ representing the rays which are reflected at the angle of incidence pOy out of every 100 rays incident.

The center c of the hyperbola is a point which is easily found from the equation; for we have $(y-a)(r+b-x)=c^2$, the equation of the rectangular hyperbola referred to the asymptotes, and the asymptote CE is distant the quantity a from Ox, whilst the other CD is distant b beyond the parallel line to Oy through A.

Now $OM = x$ is $r \sin i$, therefore

$$\text{the rays reflected of every 100 incident} = a + \frac{c^2}{b + r(1 - \sin i)}.$$

By marking the observed quantities of the reflected light upon diagrams and searching for the numbers which best represented the experiments, the author concluded that the following were *very nearly* the values for the kinds of crown, plate, and flint glass which he used.

Glass.	Specific Gravity.	Refractive Index.	Value of a.	Value of b.	Value of c^2.
Crown...	2·541	1·524	2·70	1·04	76
Plate ...	2·511	1·517	2·58	1·13	81
Flint ...	3·225	1·570	2·63	1·44	100

We obtain the number of rays reflected at the perpendicular incidence, by putting $i = 0$, to be $a + \frac{c^2}{r+b}$; so that the reflexion at a perpendicular incidence depends upon the properties which determine the values of the whole of the three constants a, b and c.

The number of rays reflected at the highest incidence is found, by putting $i = 90°$, to be $a + \frac{c^2}{b}$, and the three constants are again involved in the value.

This last is never equal to 100 the number supposed incident, as Fresnel's hypothetical formula gives it; as will be seen in the table of the values of the reflexion at different incidences at page 114, Part I.

ART. 22. The physical properties on which the values of the constants depend are still unknown to us, and it will be exceedingly interesting to find if the light which enters the medium at the highest incidence should be found for other substances to be a function of the *specific heat for equal bulks*, as the author found for the above glasses, together with speculum metal and steel, and given at page 115, Part I.

ART. 23. In the original papers in *Brewster's Edinburgh Journal of Science* the investigation of the reflective power of the second surfaces of the above kinds of glass are given; and after allowing for the quantity reflected at the first surface and absorbed by the glass, it was found to be the same as at the first surface, and is therefore represented by the above formula.

ART. 24. The reflexion of highly polished metals is of an entirely different character to that of glass; as shewn at page 115, Part I. The quantity reflected of every 100 rays incident being set off in a geometrical construction like that for glass (fig. 20), they are found to be the ordinates for a straight line, as QPR, figure 21; or y representing the number of rays reflected of 100 incident, we have the equation,

$$y = ax + b.$$

For speculum metal highly polished so that when placed near the flame of a candle the surface could not be seen, the following were found for the values of a and b, when

$$x = 100 \sin i,$$

$$a = \text{trig. tang. } 355^{\circ}\cdot 12',$$

$$b = 72\cdot 3$$

For highly polished hard steel the values were found to be for a nearly the same as for speculum metal, but for b a value nearly 10 less, or $b = 62\cdot 3$, nearly.

ART. 25. It is still desirable to determine the intensity of the light reflected by glass when the beam of incident light is polarized in any given plane. The formula above given, where the intensity is represented by the ordinate of an hyperbola, may be put into a form which will most probably be found to agree with experiments. These experiments may be made without difficulty with the comparative photometer, by remembering that we may use an analyzer in place of a polarizer for one of the two reflexions which are to be compared.

Ordinary light being considered as made up of two portions polarized in any two planes at right angles to each other, at a perpendicular incidence and at the highest incidence they will be equals; and again, when the beam is considered to consist of one half polarized in the plane of incidence and the other half polarized in the perpendicular plane, then this latter will give no intensity for the reflected beam when incident at the polarizing angle upon the reflecting surface. These data suffice to put the formula for the ordinate to the hyperbola into the following form for polarized light.

The incident beam being supposed to consist of 100 rays polarized in a plane inclined at an angle θ to the plane of incidence, and i = the angle of incidence, then

The intensity of the reflected beam

$$= \frac{1}{2}\left[a + \frac{c^2\left\{1 + \mu\sqrt{\left(\frac{r}{c^2}\right)} . \cos 2\theta . \sin 2i\right\}}{b + r\,(1 - \sin i)}\right].$$

With regard to the coefficient $\mu\sqrt{\left(\frac{r}{c^2}\right)}$, or $\frac{10\mu}{c}$, we must remember that the constants a, b, c^2 are only approximately determined from experiments.

CHAPTER IV.

ON THE INTERFERENCE OF ORDINARY LIGHT.

In the fundamental experiment of interferences for which we are indebted to M. Fresnel, the light which has radiated around a luminous point, is made by two plane mirrors slightly inclined, or by a very obtuse-angled prism, to radiate as if coming from two points near together, and the two beams thus produced on meeting, being in a fit state, interfere, producing a series of bright and dark bars, as described in Part I. page 48.

Art. 26. Prop. *To find the value of the luminiferous interval* λ *from the experiment of the two mirrors slightly inclined.*

Let a, b be the the points in fig. 22, from which the light radiates after reflexion by the two mirrors; it was shewn, in Part I. page 52, how the distance ab is determined experimentally, and therefore may be supposed to be known. Again, the distance from a and b of any point p, where the luminiferous surfaces meet and interfere, can be measured, and will therefore be known: also by means of a micrometer and eye-piece, the distance from p to q, where the interference is again of the same character as at p, can be measured in the field of view, and will be therefore again known. Now r being the succeeding place of like interference with p, we have $pr = \lambda$, and on account of its smallness compared with pq we may consider the triangle pqr as rectilineal, and the angle pqr equals the angle apb, since the luminiferous surfaces are at right angles to the rays respectively; and the triangles pqr, apb are similar;

$$\therefore \frac{ab}{pa} = \frac{pr}{pq},$$

and $$pr = \lambda = \frac{ab}{pa} \times pq.$$

If we put $ab = 2a$, $pq = m$, and $pa = l$,

we have $$\lambda = \frac{2a}{l} \cdot m,$$
as required.

Reciprocally, we have when λ is known,

$$m = \frac{l}{2a} \cdot \lambda.$$

We see that the places of like interference occur at equal distances *m* very nearly, and the breadths of the bright and dark bars are proportional to λ, and therefore are narrowest for violet and broadest for red light. The middle bright bars are in this manner coloured in ordinary light, red on their outer edge and violet on their inner edge, and the colours are symmetrical on each side of the line which bisects perpendicularly the distance *ab*.

By this symmetrical arrangement of the colours we are enabled to judge which is the middle bar, where the rays have travelled over equal spaces along the bisecting line in figure 22. In normal circumstances when the sun is high in the heavens and the atmosphere without cloud or haze, this central bar is seen black, contrary to the undulatory theory, with the colours of the bars symmetrical on each side of it. The author, to form a luminous point in an unobjectionable manner, used a *convex* mirror of speculum metal*, so that the rays from the *virtual* image of the sun had not been in a previous state of interference near a caustic.

ART. 27. PROP. *From the observed phenomena of Newton's transmitted rings, to investigate the formula for the intensity when beams of different brightness interfere.*

It was explained in Part I. page 73, how Newton's transmitted rings being produced by interfering beams of very different intensities, pointed out the conditions to which the true interference formula must be conformable.

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* *Phil. Mag.* for May, 1840, page 380.

The ratio of the intensities of the two interfering beams being the same for all colours, but very unequal (as 1 to 900 nearly), that of the compound beam was found to depend upon the colour, or upon λ; this shews that the required formula is an exponential function in λ of the retardation of one of the beams behind the other.

Taking the figure 68, Part I., or fig. 23, which is the same enlarged. Let $CA'ADBB'$ be the thin transparent film. SA a ray incident nearly perpendicularly to the film, which gives rise to a reflected ray AP and a refracted ray AB. The refracted ray gives rise to a transmitted ray BR and a reflected ray BA'. The reflected ray BA' gives rise to an emergent ray $A'Q$ and a reflected ray $A'B'$, of which the chief part is emergent at B' in $B'T$, but another part is reflected again at B', and so onwards.

Now the luminiferous surface of the incident light, of which SA is one of the rays, is perpendicular to the ray as at M, and at an after period of time gives rise to the four luminiferous surfaces at m, n, p, q, which are perpendicular to their respective rays; and the retardations mn and pq are the equivalents in air to the space $AB+BA'$ in the film, or if μ be the refractive index of the film, then

$$mn = pq = \mu(AB + BA').$$

The intensities of the beams interfering after reflexions at A and B are not greatly different; being about $\frac{1}{30}$th of the incident light in glass, that at B having suffered loss from the first reflexion at A, from absorption within the film, and the second reflexion at A' only. For these reasons the interference rings or curves are bright and black alternately in homogeneous light, from the reflected light of which AP and $A'Q$ are rays.

In the transmitted light, of which the rays have emerged at B and B', the intensities are very different; the first emergent light having its intensity but little less than that of the incident light, whilst the interfering light has been reflected both at B and A', and its intensity is $\frac{1}{30}$th of $\frac{1}{30}$th of the incident light, with a deduction for small losses. Hence the transmitted rings and curves are faint; but seen in the different parts of the solar spectrum, the bright and dark rings have very different relative intensities.

From this we find that the intensity in different parts of the solar spectrum is an exponential function in λ, of some base peculiar to each colour. The experiments described at page 76, Part I. lead us to the following forms.

Let I be the intensity of a beam of light for which the luminiferous interval is λ and the base L; then

$$I \doteq (L)^{\nu\lambda},$$

where ν is a constant depending on the system of measures employed; with the English measures in inches it is about 150,000; or if the values of λ were measured in units of $\frac{1}{150000}$th of an inch, then ν would be unity.

When we have two interfering beams, it is clear the bases must be compounded for a new base raised to the power $\nu\lambda$ for the peculiar colour of light. Now the bases compound by the law of the parallelogram as stated at page 74, Part I. the angle being in the ratio to four right angles, which $\frac{\lambda}{2}$ plus or minus the retardation, supposed less than λ, is to λ.

When the retardation is greater than λ the ratio is that of $\frac{\lambda}{2}$ plus or minus the excess of the retardation above an integral multiple of λ, to λ.

Now the contiguous angles of the parallelogram being equal to two right angles, or one the supplement of the other, then when anatropy exists we have to take the diagonal joining the other angles to those in the normal case.

If l and l' were the sides of a parallelogram inclined at an angle θ, L the diagonal from that angle, then

$$L^2 = l^2 + l'^2 + 2ll' \cos\theta.$$

These being the bases of interfering beams of light, we have the interference intensity I thus expressed:

$$I = (L)^{\nu\lambda} = \{l^2 + l'^2 + 2ll' \cos\theta\}^{\frac{\nu\lambda}{2}}.$$

Let δ be the retardation of pq in fig 23, then

$$\theta = 2\pi \frac{\left(\frac{\lambda}{2} \pm \delta\right)}{\lambda}$$

$$= \pi\left(1 \pm \frac{2\delta}{\lambda}\right),$$

and $$\cos\theta = \cos\left(\pi \pm \frac{2\pi\delta}{\lambda}\right)$$

$$= -\cos\left(2\pi\frac{\delta}{\lambda}\right);$$

$$\therefore\ I = \left(l^2 + l'^2 - 2ll' \cos 2\pi\frac{\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}}.$$

Now when $\delta = 0$, where the two surfaces of the film have no sensible interval, then

$$I = (l^2 + l'^2 - 2ll')^{\frac{\nu\lambda}{2}}$$

$$= (l - l')^{\nu\lambda},$$

and there is a dark curve, as seen in the first dark ring of Newton's transmitted rings, around the central bright spot; where the two surfaces of glass are in optical contact, and near it form only one reflecting surface, shewing a gray light.

There are dark curves again when

$$\delta = \lambda, \text{ or } 2\lambda,\ 3\lambda,\ \&\text{c}\ldots.n\lambda,$$

and n is any integer, since we have as before

$$I = (l - l')^{\nu\lambda}.$$

When $$\delta = \frac{\lambda}{2}, \text{ or } \frac{3\lambda}{2},\ \frac{5\lambda}{2},\ \&\text{c}\ldots.\frac{2n+1}{2}\lambda,$$

and n is any integer, we have

$$I = (l^2 + l'^2 + 2ll')^{\frac{\nu\lambda}{2}}$$

$$= (l + l')^{\nu\lambda},$$

and there are a series of bright curves alternating with the former dark ones.

Now the ratio of these intensities in the bright and dark curves or rings is

$$\frac{\text{intensity in a bright curve}}{\text{intensity in a dark curve}} = \left(\frac{l+l'}{l-l'}\right)^{\nu\lambda};$$

and taking the twice reflected light $\frac{1}{900}$ th part of the first emergent light, or

$$\frac{(l)^{\nu\lambda}}{(l')^{\nu\lambda}} = 900,$$

$$\frac{\text{intensity in a bright ring}}{\text{intensity in a dark ring}} = \frac{l^{\nu\lambda}\left(1+\dfrac{1}{900^{\frac{1}{\nu\lambda}}}\right)^{\nu\lambda}}{l^{\nu\lambda}\left(1-\dfrac{1}{900^{\frac{1}{\nu\lambda}}}\right)^{\nu\lambda}}$$

$$= \left(\frac{900^{\frac{1}{\nu\lambda}}+1}{900^{\frac{1}{\nu\lambda}}-1}\right)^{\nu\lambda}.$$

In red light this ratio was found to be about 3·6 by experiment.
In green light..................................2·4....................
Between the violet and indigo..................1·56..................

Taking the values of $\nu\lambda$ which produce these results, we find them to be rather more than 3·8 for the result in red, a little less than 3·4 for that in green, and about 2·7 for that in the indigo-violet.

These agree very closely in giving $\nu = 150{,}000$ for the values of λ in Dr Young's table, page 70, Part I.

ART. 28. In the reflected light, one portion has been reflected at the first surface and the other at the second surface of the film, and we have anatropy, so that the other diagonal of the parallelogram is to be employed in the composition of the bases of the interfering lights. We have now consequently the intensity I as follows:

$$I = \left(l^2 + l'^2 + 2ll' \cos 2\pi \frac{\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}}.$$

nd now $l = l'$ nearly, and if I' be the intensity of either of the flected beams which equals $\frac{1}{30}$th of that of the incident light early, we have,

$$I = l'^{\nu\lambda} \left\{2\left(1 + \cos 2\pi \frac{\delta}{\lambda}\right)\right\}^{\frac{\nu\lambda}{2}}$$

$$= I' \left\{2\left(1 + \cos 2\pi \frac{\delta}{\lambda}\right)\right\}^{\frac{\nu\lambda}{2}}.$$

When $\delta = 0$, or λ, 2λ, &c....$n\lambda$, where n is any integer, then

$$I = I'.2^{\nu\lambda},$$

nd the intensity is a maximum at the distance from C fig. 23, here the minimum intensity occurred in the transmitted light.

When $\delta = \frac{\lambda}{2}$, or $\frac{3\lambda}{2}$, $\frac{5\lambda}{2}$, &c.... $\frac{2n+1}{2}\lambda$, where n is any teger, then

$$I = 0,$$

nd there are black curves at the distances from C where the right ones occurred in the transmitted light.

The two series of reflected and transmitted curves on being uperposed in M. Arago's experiment, nearly but not entirely in ll colours, obliterate each other, and produce, as will be found n applying numerical calculations, a nearly uniform light.

When we know the values of λ and the correct formula for he intensity of interfering lights, we may easily investigate the henomena of interference in any cases.

ART. 29. *In the experiment of the two mirrors slightly inlined, to find the nature of the interference at a given point.*

Let a and b be the points from which the beams of light diate after reflexion by the two mirrors, in fig. 24.

Draw Ox the axis of x bisecting perpendicularly the distance b, and Oy the axis of y through a, O, and b.

Let $ab = 2a$, and taking any point P, let $OM = x$, $PM = y$ be its co-ordinates which are given; also a and y are supposed small compared with x.

Draw Pa and Pb, then

$$Pa = \sqrt{\{x^2 + (y \sim a)^2\}}$$

$$= x + \frac{(y \sim a)^2}{2x}, \quad \text{nearly},$$

and similarly $$Pb = x + \frac{(y + a)^2}{2x}, \quad \text{nearly}.$$

Now the retardation

$$\delta = Pb - Pa$$

$$= \frac{(y + a)^2 - (y \sim a)^2}{2x}$$

$$= \frac{2ay}{x}, \quad \text{nearly},$$

and the interference at P

$$= I = \left\{l^2 + l'^2 - 2ll' \cos 2\pi \frac{\delta}{\lambda}\right\}^{\frac{\nu\lambda}{2}}.$$

Since the beams are of equal intensity and $l = l'$, let the intensity of either $= I' = (l)^{\nu\lambda}$;

then $$I = l^{\nu\lambda} \left\{2\left(1 - \cos 2\pi \frac{\delta}{\lambda}\right)\right\}^{\frac{\nu\lambda}{2}}$$

$$= I' \left\{2\left(1 - \cos 2\pi \frac{\delta}{\lambda}\right)\right\}^{\frac{\nu\lambda}{2}}.$$

When $\delta = 0$, or λ, 2λ, &c. ... $n\lambda$, where n is any integer, then

$$I = 0,$$

and we have a series of black bars nearly equidistant, since δ varies as y, nearly.

When $$\delta = \frac{\lambda}{2}, \text{ or } \frac{3\lambda}{2}, \frac{5\lambda}{2}, \text{ \&c.} \ldots \frac{2n+1}{2}\lambda,$$

and n is any integer, then the intensity is a maximum, and

$$I = I' . 2^{\nu\lambda},$$

r there are a series of bright bars alternating with the dark nes.

For intermediate values of δ, we have

$$I = I' \left\{ 2 \left(1 - \cos 4\pi \frac{ay}{\lambda x} \right) \right\}^{\frac{\nu\lambda}{2}},$$

nd there are intermediate intensities.

In white light the bars are coloured, arising from their readths depending on λ, for each homogeneous tint; and from he mixture of colours only a few bars are distinguishable.

ART. 30. PROP. *To determine the velocity with which light raverses a plate of glass.*

The proof given by Dr Young, that the velocity of light in ny medium is inversely proportional to the refractive index, from the contraction of the radii of Newton's rings when water was placed between the lenses by which they were formed, is sufficient for a perpendicular incidence. The experiment here discussed, which was first tried by M. Fresnel, is nevertheless important from its corroborating the other method for a direct incidence, and it may in some future time serve to determine whether the same rule holds for oblique incidences. From the author's experiments* it appeared that light might be eventually found to move slower for oblique incidences than for the perpendicular one.

Let a and b (figure 25) be the luminous points of the last proposition, and let a plate of glass with accurately parallel and plane surfaces be placed at $c\,d\,g\,h$, perpendicular to the line AO which bisects perpendicularly the distance ab; then the interference will be seen in the light which has passed through the plate, nearly as if it were not there. Now if the plate be divided into two by a diamond cut ef we may turn one of these pieces into a new position fk, and the interference which was seen originally at A will be found to have descended as the plate fk was gradually inclined to a place A'. Now the time of traversing the distances in air and glass from a and b to A, being the same, it will be so also in regard to traversing the distances in air and glass to A'. Let m and m' be the distances in air, S and S' the distances in glass travelled from a and b respectively to A', which are to be determined from the measures in the experiment. Let v be the velocity of light in air, u that in the upper piece of glass, u' that in the lower. Then since the velocities are uniform,

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* *Phil. Mag.* for 1833, Vol. III. p. 333.

$$\text{time} = \frac{\text{space described}}{\text{velocity}}, \quad \text{and we have}$$

$$\frac{m}{v} + \frac{S}{u} = \frac{m'}{v} + \frac{S'}{u'}.$$

Now, if the velocity of light in glass is the same at all incidences, we have $u = u'$, and the above equation will determine u. At a perpendicular incidence very nearly the experiments agree with the other method in giving $u = \frac{v}{\mu}$, but they leave room to suppose that u' may for high incidences be sensibly different from $\frac{v}{\mu}$.

These experiments require great nicety and good instruments.

ART. 31. PROP. *To investigate the radii of Newton's rings when formed between spherical surfaces of given curvatures.*

The thin film of air being formed between spherical surfaces of given radii, the thickness of the film at any given point is easily expressed.

Let ACB (figure 26) be the thin film of air between the spherical surfaces of the lenses which are in contact at C, with the common normal OCO' passing through O and O', the centers of curvatures of the surfaces. The interference is that of the rays arising from a ray SA incident nearly perpendicularly, having the retardation mn in the reflected rays equal to pq in the transmitted rays, and equal to twice AB the thickness of the plate of air.

Now drawing AM, BN perpendicularly upon OCO', we have

$$AB = CM + CN.$$

Let r = radius of the upper spherical surface,

r' = ... lower ....

and $R = MA$ or NB the radius of the interference ring round C.

Since CM and CN are the versed sines of the arcs CA and CB respectively, we have

$$CM = \frac{AM^2}{2\,.\,OC} = \frac{R^2}{2r}, \qquad \text{nearly,}$$

$$CN = \frac{BN^2}{2\,.\,O'C} = \frac{R^2}{2r'}, \qquad \text{nearly,}$$

and the retardation δ of the interfering rays $= 2\,.\,MN$,

$$\text{or} \qquad \delta = R^2\left(\frac{1}{r} + \frac{1}{r'}\right).$$

If the retardation occurred in a beam incident obliquely, we should have it of another form, as shewn in the next article.

Substituting the value of δ in the formula for the intensity I in the reflected rings, we have

$$I = \left\{l^2 + l'^2 + 2ll' \cos 2\pi \frac{R^2}{\lambda}\left(\frac{1}{r} + \frac{1}{r'}\right)\right\}^{\frac{\nu\lambda}{2}},$$

or, since $l = l'$, if I' be the intensity of the reflected light at A,

$$\text{then} \qquad I = I'\left[2\left\{1 + \cos 2\pi \frac{R^2}{\lambda}\left(\frac{1}{r} + \frac{1}{r'}\right)\right\}\right]^{\frac{\nu\lambda}{2}},$$

and the bright rings occur where

$$R^2\left(\frac{1}{r} + \frac{1}{r'}\right) = n\lambda,$$

where n may be any integer; and with a bright center where $n=0$,

$$R^2 = \frac{n\lambda r\,.\,r'}{r + r'},$$

which gives the radius R of the bright ring, when n and λ are given. In the same way the radii of the black rings are determined from the expression

$$R^2\left(\frac{1}{r} + \frac{1}{r'}\right) = \frac{2n+1}{2}\,.\,\lambda,$$

$$\text{and} \quad R^2 = \frac{(2n+1)\,\lambda \,.\, r \,.\, r'}{2\,(r+r')}.$$

Giving n the series of values 1, 2, 3, &c., we have the radii of the bright rings proportional to the series of numbers

$$1,\ \sqrt{2},\ \sqrt{3},\ \&c.,$$

and the radii of the black rings to the series of numbers

$$1,\ \sqrt{3},\ \sqrt{5},\ \sqrt{7},\ \&c.,$$

as found by experiment.

Similarly in the transmitted rings with a dark center, the radii of the dark rings are proportional to the terms of the series

$$1,\ \sqrt{2},\ \sqrt{3},\ \&c.,$$

and the radii of the bright ones to the terms of the series

$$1,\ \sqrt{3},\ \sqrt{5},\ \&c.$$

ART. 32. PROP. *To investigate the interference phenomena of thin plates when the obliquity is considerable.*

Let $CA'ADBB'$ in figure 23 be the thin film as before, and let the letters refer to points analogous to those in the previous propositions, and draw Bb the perpendicular thickness t of the film at B.

Let Am be the distance in air of a luminiferous surface from A, when the one from the reflexion at B arrives at A'; then i being the angle of incidence, i' that of refraction, we have

$$\begin{aligned} Am &= \mu\,(AB + BA') \\ &= 2\mu t \sec i', \end{aligned}$$

since the surfaces may be considered parallel for the small space AA'.

$$\text{And} \qquad An = AA' \sin i = 2t \tan i' \,.\, \sin i;$$

$$\begin{aligned} \therefore \text{ the retardation } \delta &= Am - An \\ &= 2t\,(\mu \sec i' - \tan i' \,.\, \sin i) \\ &= 2t\mu \left(\frac{1 - \sin^2 i'}{\cos i'}\right) \\ &= 2t\mu \cos i'. \end{aligned}$$

This is the formula for the retardation in the colours seen obliquely on the soap bubble and other thin transparent films. It becomes

$$\delta = 2t \cos i$$

for the thin plate of air between surfaces of glass or other transparent bodies, since for air $\mu = 1$. And i is put for the angle between the ray and the normal to the surfaces. The thickness and the forms of the outer surfaces of the lens or prism do not affect the interference due to the thin plate of air, on account of the relation between the velocity in the medium and the refractive index.

ART. 33. PROP. *To investigate the measures of Newton's rings seen at considerable obliquities.*

In Art. 31 the perpendicular thickness t of the plate of air between the lenses near the point of contact C is MN in fig. 26, and therefore

$$\delta = R^2 \left(\frac{1}{r} + \frac{1}{r'}\right) \cos i.$$

Now when δ is constant for any given ring, we have

$$R^2 = \frac{rr'.\delta \sec i}{r + r'},$$

and R varies as $\sqrt{(\sec i)}$.

Newton found the experimental result not quite the same as this, and he has given the rule accordingly*; but it has been usual to consider the reflected rings as commencing with a black central spot, and the reaction of the two reflective surfaces upon each other near the point of contact may produce the difference observed. In the theory again the surfaces of the lenses are considered the reflecting surfaces, which may not be quite accurate, as the reflective power of a surface depends evidently upon other properties than the simple extent† of the dense particles of bodies.

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* *Opticks*, 3rd ed. p. 180; Herschel's *Treatise on Light*, Art. 639.

† See *Report of the first Meeting of the British Association at York, in* 1831, p. 74.

On Diffraction.

Since it can be easily demonstrated by experiment, as described at page 56, Part I., that a considerable and visible quantity of light is reflected from the sharpest edge to which any substance can be brought, and is also inflected into the shadow, we have sufficient and evident reasons for the interference of light which is produced by diffraction.

ART. 34. PROP. *To investigate the interference near the geometrical shadow of a body placed in the light which radiates from a luminous point.*

Let O be the luminous point, A the edge of an opaque body, CPB a screen, or the field of view of an eye-lens with which the diffraction fringes are seen.

Let the straight line OA meet the screen perpendicularly at B, then B is the boundary of the geometric shadow of the body. The light inflected into the shadow will gradually shade away to darkness, but that reflected from the edge A meeting the direct light from O will interfere with it.

Let P be any point on the screen, and $PB = x$, which is small in the experiments, compared with OA and AB.

Let $OA = a$, $AB = b$, and δ the retardation of the ray which has been reflected at A in AP, behind the direct ray OP.

$$\begin{aligned}
\text{Then}\qquad \delta &= (OA + AP) - OP \\
&= a + \surd(b^2 + x^2) - \surd\{(a+b)^2 + x^2\} \\
&= a + b + \frac{x^2}{2b} - (a+b) - \frac{x^2}{2\,.\,(a+b)}, \qquad \text{nearly,} \\
&= \frac{x^2}{2}\left(\frac{1}{b} - \frac{1}{a+b}\right) \\
&= \frac{x^2}{2}\cdot\frac{a}{b\,(a+b)}.
\end{aligned}$$

The intensity I at the point P is given by the formula

$$I = \left(l^2 + l'^2 - 2ll' \cos 2\pi \frac{\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}}.$$

Now the light reflected at A being small for a sharp edge, compared with the direct light, therefore the difference of the maxima and minima given by the formula will never be large, and the bars seen parallel to the edge of the body are faint.

The minima occur when $\delta = 0$, or λ, 2λ, &c. ... $n\lambda$, and n is any integer. Or at B and at a succession of points where

$$n\lambda = \frac{x^2}{2} \cdot \frac{a}{b(a+b)},$$

or where $$x = \sqrt{\left\{\frac{2n\lambda b\,(a+b)}{a}\right\}},$$

and giving n the successive values 1, 2, 3, &c., the distances of the dark bars from B are proportional to the series

$$1,\ \sqrt{2},\ \sqrt{3},\ \&c.,$$

and the bars are closer as they are further from B, as is easily seen in the experiment.

The maxima of brightness occur when

$$\delta = \frac{\lambda}{2},\ \text{or}\ \frac{3\lambda}{2},\ \frac{5\lambda}{2},\ \&c. \ldots \frac{2n+1}{2}\lambda;$$

and from $$\frac{2n+1}{2}\lambda = \frac{x^2}{2} \cdot \frac{a}{b(a+b)},$$

giving n the series of values 0, 1, 2, &c., we have the series of bright bars in distance from B following the series

$$1,\ \sqrt{3},\ \sqrt{5},\ \&c.$$

We see that these bars follow the analogy of Newton's transmitted rings, counted from a black center.

The distance x of any bar depending on $\sqrt{\lambda}$, the bars are narrower in violet and broader in red light, and hence they are strongly coloured, and only a few are visible in white light. The distance x of a given bar depends also on the values of a and b.

ART. 35. PROP. *To investigate the interference within the geometrical shadow of a narrow body placed in the light radiating from a luminous point.*

Let O be the luminous point, AB the narrow body whose breadth is $2a$. Draw the straight line OcC bisecting AB perpendicularly. Let PC perpendicular to OcC be the screen on which the interference is to be examined, and $Cc = b$, $PC = x$. Now outside the geometric shadow determined by drawing lines from O past A and B, the interference is the same as that for a single edge investigated in the last article; therefore let P lie between these lines, as in figure 28, and the interference then arises from the light inflected at A and B into the shadow.

The retardation δ of the ray PB behind PA is

$$\begin{aligned}\delta &= PB - PA\\ &= \sqrt{\{b^2 + (a+x)^2\}} - \sqrt{\{b^2 + (a-x)^2\}}\\ &= b + \frac{(a+x)^2}{2b} - b - \frac{(a-x)^2}{2b}, \qquad \text{nearly},\\ &= \frac{2ax}{b}.\end{aligned}$$

Let I be the intensity of the light at P, then since we have anatropy*, and $l = l'$ for the equal bases of the inflected lights nearly,

$$\begin{aligned}I &= \left(l^2 + l'^2 + 2ll' \cos 2\pi \frac{\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}}\\ &= l^{\nu\lambda}\left\{2\left(1 + \cos 4\pi \frac{ax}{b\lambda}\right)\right\}^{\frac{\nu\lambda}{2}}.\end{aligned}$$

The intensity is never very great from the small value of l for the inflected light. The maxima of brightness occur when

$$\delta = 0, \text{ or } \pm\lambda, \ \pm 2\lambda, \text{ \&c.} \dots \pm n\lambda,$$

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* See Part I. p. 57.

where n is any integer; or there is a central bright bar, and a series of other bright bars on each side of it, for which

$$x = \pm \frac{bn\lambda}{2a},$$

and giving n the series of values 1, 2, 3, &c., we see that the bars are at equal distances, unlike those outside the geometric shadow.

When δ has the values $\pm \frac{\lambda}{2}, \pm \frac{3\lambda}{2}, \pm \frac{5\lambda}{2}$, &c.... $\pm \frac{2n+1}{2}\lambda$,

then the intensity is zero, or

$$I = 0,$$

and a series of black bars alternate with the bright ones.

The central bar, where $\delta = 0$, being a bright bar, whatever be the value of λ, is white; but the ones on each side of it in ordinary light are coloured violet on their inner edges and red on their outer edges, since the places of their maxima depend upon λ.

Cor. If the small body, instead of being a narrow parallelogram, is a small circular disc, the same investigation shews that there is a central bright spot with black and bright rings within the shadow, around it.

Art. 36. Prop. *To shew that the bars of the last proposition at different distances from the narrow body follow the course of hyperbolic arcs.*

This solution applies to other cases where light from two luminous points interferes, as for instance the light reflected by two mirrors slightly inclined. The point P being anywhere within the geometric shadow, let c be the origin of co-ordinates, and $Cc = y$, $PC = x$ the co-ordinates of P. Now δ being constant for a given bar, putting y for b the expressions of the last article become

$$\delta = \sqrt{\{y^2 + (a + x)^2\}} - \sqrt{\{y^2 + (a - x)^2\}},$$

or $$\delta + \sqrt{\{y^2 + (a - x)^2\}} = \sqrt{\{y^2 + (a + x)^2\}},$$

from which we find

$$x^2\left(\frac{4a^2}{\delta^2}-1\right)-y^2-\left(a^2-\frac{\delta^2}{4}\right)=0,$$

which is the equation of an hyperbola.

ART. 37. PROP. *When light from a very distant luminous point passes through an aperture in a thin opaque plate, which is a very narrow parallelogram, to find the phenomena at any point upon the screen.*

Let BC be the perpendicular section, in fig. 29, of the *very narrow* parallelogram which is an aperture in an opaque plate, SA, SB, SC parallel rays of the pencil, of which SB and SC are diffracted at the edges of the parallelogram.

Let SAD be any line perpendicular to the plate and screen PD.

Now P being a point where rays from B and C interfere, let $AB = x$, $AC = x'$, and $PD = y$; let also $AD = a$, which is large compared with x and y.

To find the retardation $\delta = BP - CP$, we have

$$BP^2 = a^2 + (y-x)^2,$$

$$CP^2 = a^2 + (y-x')^2,$$

$$\text{and} \qquad \begin{aligned} BP^2 - CP^2 &= (BP - CP)(BP + CP) \\ &= \delta\,(BP + CP) \\ &= 2y\,(x'-x) + x^2 - x'^2 \\ &= 2\,(y-x)\,(x'-x), \qquad \text{nearly.} \end{aligned}$$

When $(x'-x)$ is small, say $\frac{1}{200}$th of an inch or smaller, and x very small compared with y, we have

$$\begin{aligned} \delta &= \frac{2y\,(x'-x)}{BP + CP} \\ &= \frac{y\,(x'-x)}{a}, \qquad \text{nearly,} \end{aligned}$$

for a first approximation.

Applying this value of δ in the formula for the intensity I, we have

$$I=\left\{l^2+l'^2+2ll'\cos 2\pi\frac{y\,(x'-x)}{a\lambda}\right\}^{\frac{\nu\lambda}{2}},$$

which depends upon the breadth $(x'-x)$ of the aperture, but not on the value of x supposed small compared with y, so that there may be a great number of such apertures, narrow and close together, which will satisfy the conditions, and the effect at P is the superposition of like results, and in the maxima produces considerable brilliancy. The minima will be nearly black, whilst l and l' are nearly equal.

The maxima occur when

$$\frac{y\,(x'-x)}{a}=0, \text{ or } \lambda,\ 2\lambda,\ 3\lambda,\ \&\text{c.}\dots n\lambda;$$

and n any integer or a maximum occurring at D when $y=0$, we have a succession of maxima at the equal distances, of which the first is

$$y=\frac{a\lambda}{x'-x};$$

or using the angular measure with center A,

$$\frac{y}{a}=\frac{\lambda}{x'-x},$$

this formula explains the origin of Fraunhofer's experimental laws* as follows, the experiment being made with a telescope, as explained in Art. 42.

"1. *The angles of deviation of the diffracted rays, forming similar points of the systems of fringes produced by different apertures, are inversely as the breadths of the apertures.*

"2. *That the distances of similar rays (the extreme red, for instance,) from the middle in the several spectra, constituting the successive fringes, form in each case an arithmetical progression whose difference is equal to its first term.*

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* *Denkschriften der Akademie zu München für das Jahr* 1821, and Herschel's *Treatise on Light*, Art. 741.

"3. That calling γ the breadth of the aperture, in fractions of a Paris inch, the angular distances L', L'', L''', &c., in parts of a circular arc to radius unity, of the extreme red rays in each fringe from the middle line, are respectively represented by $L' = \frac{L}{\gamma}$, $L'' = \frac{2L}{\gamma}$, $L''' = \frac{3L}{\gamma}$, &c., where $L = 0{\cdot}0000211$, and a similar law holds for all the other coloured rays, different values being assigned to L for each."

It is clear that Fraunhofer's laws are only approximative, which was shewn also by his finding the colours mixed.

ART. 38. PROP. *When light from a luminous point passes through an aperture in a thin opaque plate, which is a long parallelogram and not very small, to find the illumination on a screen at a given point.*

Let O be the luminous point in fig. 30, AB the section of the aperture perpendicular to the longer sides of the parallelogram. Draw OAM, OBN, then between M and N the direct light falls upon the screen, and there is interference between it and the light reflected at the edges A and B; but outside M and N the rays reflected from one edge interfere with those inflected from the other.

The luminiferous surface at the screen will have the form $kmCnl$*, and between M and N we shall have the two reflected rays interfering with the direct rays and with each other; this involves more complicated rules for the interference than we have considered, especially as the law of the relative intensities of the reflected lights is unknown.

Draw OcC bisecting the distance AB in c, and meeting the screen perpendicularly at C. Take P any point where the interference takes place. Let l be the base of the direct light at P, l' and l'' those of the lights reflected at A and B respectively. Then if L is the base at P from compounding l, l', and θ the angle between them, we have

$$L^2 = l^2 + l'^2 + 2ll' \cos\theta.$$

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* Part I. p. 54.

To compound L with l'' we require the angle between l'' and L. Now taking ϕ the angle between L and l, fig. 74, Part I., we shall have

$$\sin\phi = \frac{l'}{L}\sin\theta,$$

which is small when l' is small compared with l, and we may approximately take the angle θ' between L and l'', as between l and l'', whilst l is large between M and N; but outside M and N, $l = 0$, and the angle θ' is that between l' and l''.

Then if L' is the base compounded from L and l'', or l, l', and l'', we have

$$\begin{aligned} L'^2 &= L^2 + l''^2 + 2Ll''\cos\theta' \\ &= l^2 + l'^2 + l''^2 + 2ll'\cos\theta \\ &\qquad + 2\sqrt{(l^2 + l'^2 + 2ll'\cos\theta)}\,.\,l''\cos\theta' \ \ldots\ldots\ldots\ (1), \end{aligned}$$

and neglecting l' under the sign of the square root, as small compared with l, we have approximately

$$L'^2 = l^2 + l'^2 + l''^2 + 2l\,(l'\cos\theta + l''\cos\theta') \ \ldots\ldots\ldots\ (2),$$

and the intensity at P is

$$I = (L')^{\nu\lambda},$$

$$\text{also} \qquad \theta = \pi \pm \frac{2\pi\delta}{\lambda}, \qquad \theta' = \pi \pm \frac{2\pi\delta'}{\lambda}.$$

Between M and N, δ and δ' have the values determined as in Art. 34. For points beyond M and N, neglecting l in the value of L', we have

$$L'^2 = l'^2 + l''^2 + 2l'l''\cos\theta' \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3),$$

$$\text{and} \qquad I = (L')^{\nu\lambda},$$

where θ' must be determined as in Art. 37.

The results will be as follows: along the middle line cC there will be interference of the character such as would arise from a single edge, but brighter in the maxima and darker in the minima, as shewn by taking $l' = l''$ and $\theta = \theta'$ in (2).

On each side of C as θ and l'' diminish, θ' and l' increase, and the effect is modified from that of a single edge, depending on the distance of the screen from AB and the position of P between M and N. When the screen is near AB and P near M or N, the effect becomes that of a single edge again.

When P is outside M or N, putting $l=0$ in (1), we have as in (3),

$$L'^2 = l'^2 + l''^2 + 2l'l'' \cos\theta',$$

the result of Art. 37, and the results are the same.

We see from this discussion how the asymptotes of the hyperbolas in fig. 56, Part I., are the boundaries of the geometric shadows of the edges of the angular aperture and the line perpendicular to one bisecting the angle between them, through the point of meeting.

The hyperbolic form of the fringes outside the points M and N is easily deduced from the approximate results of Art. 37, since we have

$$y\,(x'-x) = an\lambda,$$

and the breadth $x'-x$ of the angular aperture and its shadow is proportional to the distance from the angle; let this distance equal z,

then $\quad y\,.\,z = \text{constant for a given fringe},$

and this is the equation of an hyperbola referred to the asymptotes as axes of co-ordinates.

ART. 39. PROP. *When light from a luminous point passes through a small circular hole, to find the result on a screen.*

Using the last figure, let the line OC pass through the center of the aperture; the interference at the center C will be variable as in the last proposition, and we have a central spot as described at page 59, Part I. Near the central spot the interference is determined as in the last article, and we have a series of varying rings around it.

ART. 40. *When light from a very distant luminous point passes through a grating consisting of a number of very narrow equal parallelograms, set parallel to each other at equal distances, to find the interference phenomena.*

Let $AbcBC$ be the section of the grating, on which a pencil of parallel rays falls directly, as in figure 31.

Let PD be the screen parallel to the grating. We shall have the interference as discussed in Article 37 from the opposite edges of the parallelogram as C, B, and c, b, &c.; and Fraunhofer calls the maxima or bright bars *spectra of the first class.*

We shall have also interference of the light inflected or reflected from the like edges of the parallelograms as b, B, and c, C, giving *spectra of the second class;* and the distance $BC=x'-x$ of Article 37 must be replaced by bB or cC, or the breadth of the parallelogram plus the breadth of the separation;

$$\text{let} \quad bB=cC=x''-x;$$

we have as in Art. 37, if $AD=a$, $PD=$ distance of the first maximum $=y$, then

$$\frac{y}{a}=\frac{\lambda}{x''-x},$$

which, from the larger denominator, is smaller than the former; or the *spectra of the second class* are closer together than those of the first class.

By the interference of the light reflected at an edge b with that inflected from another as C, *spectra of the third class* would arise closer together still, as observed by Fraunhofer.

We should expect the way in which the grating was formed to influence the brilliancy which any class would have with a very great number of apertures. A grating formed with wires of $\frac{1}{500}$th of an inch in diameter, arranged at double that distance from each other, would furnish a reflected light stronger than the inflected light. The reverse would arise if a like grating were formed with gold-leaf upon glass.

Fraunhofer, with his fine gratings produced by lines drawn upon glass with a diamond up to the number 8200 in an inch, found the *spectra of the second class* to become so pure that the fixed lines of the solar spectrum were well seen in them. A pure diffraction spectrum thus formed, is called *the normal spectrum*, to distinguish it from those produced with prisms, which depend upon the substance of which the prism is made, whilst the diffraction spectrum depends on the values of λ only.

The manner in which the spectra of the second class become so pure as to shew the fixed lines, is still a subject well worthy examination. It probably arises from the like reflexions of the interfering rays.

ART. 41. PROP. *The light from a luminous point, which is not so distant that the rays can be considered parallel, passes through a narrow parallelogram in a thin plate; to determine the expression for the interference at a given point.*

Let O be the luminous point, BC the section of the aperture in figure 32, P any point on the screen PD. Let OAD be perpendicular to the plate and the screen; and

$$OA = b, \quad AD = a, \quad AB = x, \quad AC = x', \quad PD = y.$$

Then the retardation δ of the ray OBP behind OCP is

$$(OB + BP) - (OC + CP),$$

or following the method of Article 37, we have

$$\delta = (x' - x)\left(\frac{y}{a} - \frac{x}{b}\right), \qquad \text{nearly},$$

and δ is not now independent of x, which must be remembered in making experiments.

Diffraction with a Telescope.

ART. 42. It has been shewn that the diffracted portion of a luminiferous surface radiating around a luminous origin O, figure 33, would after passing a straight opaque edge of a plate A, have its section perpendicular to the edge of the form pBP. In the section parallel to the edge A its form would be unchanged.

If now a telescope be directed towards O, of which C is the object-glass in figure 34, and e the eye-glass, adjusted to view distinctly the image of the luminous point O, as at q, the common focus of the lenses C and e; then the rays which pass through the object-glass at a distance from the edge A, when placed before the object-glass and partly covering it, will be brought to a focus at q as a bright point; but the rays which have been iffracted at the edge A will not be brought to a focus at q, as a ens does not affect the divergence of a pencil diverging from a oint at the lens. This is seen from the discussion of the formula f geometrical optics* $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$, where f is the focal length of lens, u and v are the distances of the conjugate foci from it; ow when $u=0$, we must have $v=0$ also; so that the rays iverging from the point A, figure 34, supposed at the lens, will fter emergence from it diverge from A still.

In this way the diffracted light is acted upon parallel to the dge, like the rays forming the image q; but perpendicular to the dge its radiation from the edge is unaffected, and hence it is ollected into a line aqb passing through the image at right ngles to the edge A.

ART. 43. If an aperture in an opaque plate, which is *bounded by straight lines*, be placed before the object-glass, the bright image of O at q has a ray like ab through it, for *each side f the polygon*, which is the aperture.

When a triangular aperture, which is *small* in proportion to the focal length of the object-glass, is placed before it, we have the six-rayed star of figure 61, Part I.; and the light in the rays interfering, we see the rays cut off from the center.

When a long and narrow aperture which is a parallelogram, in an opaque plate, is placed before the object-glass, the bright rays as ab, fig. 34, from the two sides coincide in position and direction, but their light interferes, giving a set of short lines with dark interruptions between them, to make up the longer lines of rays.

When the aperture is a small square, we have a rectangular cross through the bright star-like center, which is broken up into portions shewing bright and dark spaces by interference, and the light passing near the angles modifies the phenomena also.

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* Potter's *Optics*, Part I. p. 80.

When the telescope in these experiments is out of focus, there is a more complicated interference, as described in Part I. page 63.

ART. 44. When the object-glass of a telescope is covered except a *small circular* aperture, the image of a luminous point is surrounded by coloured rings which become broader as the diameter of the aperture is smaller in proportion to the focal length. It is immaterial on what part of the object-glass the aperture is left; so that if we have several equal circular apertures, they give only one image and set of rings, but of brighter light than if there were only one such aperture. The cause of the appearances is easily seen by combining the effect of the object-glass with the diffraction arising from the edges of the aperture, as discussed in Art. 39; the object-glass refracting the rays which pass through the central part of each aperture in the regular manner, as if there were no diffraction, and the interference taking place symmetrically around these lines.

The spurious discs of the fixed stars shewn by telescopes are the diffraction central spots arising from the edges of the object-glasses covering the true image of the star. The discs and the rings seen around them increase in size as the aperture of the object-glass is diminished.

On the Interference of Light near a Caustic.

When Huygens had discussed the forms of the luminiferous surfaces in calcareous spar, he traced also the forms which the luminiferous surfaces would assume after reflexion and refraction at spherical surfaces; but the discovery of interference as an essential property of light was reserved for Hooke and Young; and the discovery of the interference near a caustic after reflexion or refraction fell to the author's good fortune. This case of interferences has proved an important experiment from its furnishing a decisive test of the truth of theories of light, the experiment shewing results at variance with the fundamental principle of the undulatory theory.

ART. 45. *To find geometrically the form of the luminiferous surface after a diverging pencil has been reflected by a concave spherical mirror.*

Let Q be the luminous point, BAB' the spherical mirror whose center is O, and q the focus conjugate to Q in fig. 35.

Let aqa' be the caustic curve, then when the central part of the luminiferous surface arrives at m after reflexion, that surface will consist of portions ba, ama', and $a'b'$ in the figure having an edge of regression at the caustic surface, or in the section of the figure, cusps at a and a' upon the caustic. The branches ab, ama', and $a'b'$ are involutes of the caustic, and orthogonal trajectories of the system of rays. As the luminiferous surfaces are perpendicular at each point to the rays radiating from Q as indicated by the circular arcs drawn with center Q (fig. 35), so also after reflexion they are perpendicular to the reflected rays.

The forms of the surfaces after reflexion are easily drawn geometrically as follows. Let MM' be a circular arc representing the luminiferous surface at a given instant, with center Q, if the mirror had not been interposed. Taking successive points in the mirror AB, draw circles touching the arc MM', then their ultimate intersections in the reflected light give the orthogonal trajectory $bama'b'$, which is the luminiferous surface required; the distance of Q to the mirror plus that to the trajectory being constant and equal to QM as required. The luminiferous surface is thus an involute of the caustic, as described at page 79, Part I., in any of its successive positions, having cusps at the caustic as when the central part has arrived at n, until it has passed the geometrical focus q, when it becomes a curve of continued curvature.

ART. 46. PROP. *To investigate the equations for finding the form of the luminiferous surface, after reflexion at a concave spherical mirror.*

Let Q be the luminous origin (fig. 36), Lp' the place where the luminiferous surface would have arrived if the mirror BAB' had not been interposed; $pama'b$ the luminiferous surface after reflexion. Let O the center of curvature of the mirror be the origin, and Ox, Oy the axes of co-ordinates. Let P be any point in the mirror, and $OM = x'$, $PM = y'$, its co-ordinates; let the radius $OA = a$, and $QO = b$.

With center P describe a circle, as in figure 36, touching the circular arc Lp' in p', and let x, y be the co-ordinates of any point in it. Let $QL = Qp' = r$. Then

$$x'^2 + y'^2 = a^2 \text{..........................(1)},$$

$$(x' + b)^2 + y'^2 = QP^2,$$

$$Pp' = Qp' - QP = r - \sqrt{\{(x' + b)^2 + y'^2\}},$$

and the equation of the circle drawn with center P is

$$(x' - x)^2 + (y' - y)^2 = Pp'^2 = [r - \sqrt{\{(x' + b)^2 + y'^2\}}]^2 \text{... (2)}.$$

Now when x and y are the co-ordinates of p and p', the points of intersection of two consecutive circles, we have x' and y' varying from the point P to the neighbouring point, whilst x and y remain constant; therefore differentiating in respect of x' and y', we find, after substituting for $\frac{dy'}{dx'}$ its value $-\frac{x'}{y'}$, as follows:

$$\{y'(x + b) - x'y\}\sqrt{\{(x' + b)^2 + y'^2\}} = rby' \text{.........(3)};$$

and by eliminating x' and y' from the equations (1), (2), (3), we should have the equation required; which belonging to both the enveloping surfaces which are the loci of p and p' will consist of two factors which are their equations. Now the equation of Lp' is

$$(x + b)^2 + y^2 - r^2 = 0;$$

therefore dividing the above-named equation by this, we should have the quotient the equation of the involute of the caustic *pama'b* as required.

COR. To find the involute of the caustic after refraction at a spherical surface, we have to replace the equation (2) by the following, as seen by figure 37:

$$(x' - x)^2 + (y' - y)^2 = \frac{Pp'^2}{\mu^2}$$

$$= \frac{[r - \sqrt{\{(x' - b)^2 + y'^2\}}]^2}{\mu^2},$$

$$\text{and} \quad x'^2 + y'^2 = a^2.$$

The remainder of the solution is similar to that for the reflexion at a spherical surface.

ART. 47. PROP. *To find the phenomena of the interference near a caustic formed in the light reflected by a spherical mirror.*

In investigating the interference near a caustic we may suppose it to coincide with its circle of curvature, for the space under consideration; and the two branches of the section of the luminiferous surface will thus, near the cusp, be represented by two arcs of the involutes of a circle.

Let figure 38 represent this circle of curvature of the caustic with center O; let a be the cusp formed by the two arcs of involutes ap, aq. Let r be the radius of the circle, draw the tangents PpT at P, and QqT at Q, meeting the radius through a produced at T, and let $\alpha = \angle POa$; then the distance

$$Pp = \text{arc } Pa = r \,.\, \alpha,$$

$$\text{and} \quad PT = \text{tangent of the arc } Pa = r \,.\, \tan \alpha;$$

$$\therefore \; pT = r\,(\tan \alpha - \alpha)$$

$$= r\left(\frac{\alpha^3}{1\,.\,3} + \frac{2\alpha^5}{1\,.\,3\,.\,5} + \&\text{c.}\right).$$

$$\text{Also} \quad aT = r\,(\sec \alpha - 1) = r\,\frac{\alpha^2}{1\,.\,2} \quad \text{nearly};$$

$$\therefore \; \alpha = \sqrt{\left(\frac{2\,.\,aT}{r}\right)},$$

$$\text{and} \quad pT = \frac{r\alpha^3}{3} \quad \text{nearly},$$

$$= \frac{r}{3}\left(\frac{2\,.\,aT}{r}\right)^{\frac{3}{2}}$$

$$\propto (aT)^{\frac{3}{2}}.$$

Now $2\,.\,pT$ is probably the correct experimental retardation δ of the two interfering branches near the cusp, which therefore varies nearly as the distance from it raised to the power $\frac{3}{2}$. There is however room for consideration and experimental examination whether this value should not be resolved into the direction of the tangent at a to which the interfering rays are equally inclined, if the actual retardation were calculated from $pT = r(\tan\alpha - \alpha)$, or the retardation should equal

$$2pT \,.\, \cos\alpha = 2r(\sin\alpha - \alpha\cos\alpha),$$

when the distance from a was not very small.

In the interference formula

$$I = \left(l^2 + l'^2 - 2ll' \cos 2\pi\frac{\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}},$$

we have $l = l'$, since the intensity is the same in the two branches, and

$$I = l^{\nu\lambda}\left\{2\left(1 - \cos 2\pi\frac{\delta}{\lambda}\right)\right\}^{\frac{\nu\lambda}{2}}.$$

Then $I = 0$ when $\delta = 0$, or when $\delta = \lambda$, or 2λ, 3λ, &c. ... $n\lambda$, where n is any integer.

Again $$I = (2l)^{\nu\lambda},$$

when $$\delta = \frac{\lambda}{2}, \text{ or } \frac{3\lambda}{2}, \frac{5\lambda}{2}, \text{ \&c.} \ldots \frac{2n+1}{2}\lambda;$$

also intermediate values of δ give intermediate intensities.

There are consequently a series of maxima, alternating with darkness, and between them intermediate brightness; and commencing with darkness at the caustic, where $\delta = 0$.

In the experiment with the luminous point formed by the sun shining upon a minute globule of mercury, which is placed in the nearer focus of a spherical mirror of large proportional aperture, the interference is seen in brilliant and black rings, the outer bright one shading from the maximum brightness to darkness at the caustic; and the inner ones becoming closer and narrower from the law of the retardation as they are further from the caustic.

Cor. Similar results occur near the caustic formed by refraction; these are not unfrequently met with accidentally, and they can be distinguished from the diffraction bars and curves by the dark spaces being absolutely black, whilst the bright spaces are very bright.

The Rainbow.

The phenomena of the rainbow are not accurately discussed otherwise than as interference near a caustic. The caustic is formed by light which has been refracted at incidence and emergence from a drop of water, and has been reflected one or more times within the drop, and it has two branches which have a common asymptote.

ART. 48. PROP. *To find the asymptote of the caustic when a pencil of parallel rays is incident upon a refracting sphere, and the emergent rays have been reflected a given number of times within it.*

Let O be the center of the refracting sphere; SA a ray incident at A, reflected within the sphere at B, C, &c., and emergent at D. Let i be the angle of incidence which equals the angle of emergence, i' the angle of refraction at A which is also the angle of incidence and reflexion at B, C, &c. since the triangles AOB, BOC, &c. are isosceles.

Now the deviation at A and D is $(i-i')$, and at B, C, &c. is $180^\circ - 2i'$. Therefore if the emergent ray DR has been n times reflected within the sphere, its deviation D after emergence is

$$\begin{aligned} D &= 2(i-i') + n(180^\circ - 2i') \\ &= 2\{i + n \,.\, 90^\circ - (n+1) \,.\, i'\}; \end{aligned}$$

we have the relation of i and i' in the equations

$$\sin i = \mu \sin i';$$

$$\therefore \frac{di'}{di} = \frac{\cos i}{\mu \cos i'};$$

and when the deviation is a minimum in respect of the varying value of i, we have

$$\frac{dD}{di} = 0,$$

$$\text{or,}\quad 0 = 1 - (n+1)\,\frac{di''}{di},$$

$$0 = 1 - (n+1)\,\frac{\cos i}{\mu \cos i''};$$

$$\therefore\ \mu^2(1 - \sin^2 i'') = \mu^2 - \sin^2 i$$

$$= (n+1)^2(1 - \sin^2 i);$$

$$\therefore\ \sin^2 i = \frac{(n+1)^2 - \mu^2}{(n+1)^2 - 1}.$$

Having calculated the value of i from this expression, and then that of i' from $\sin i = \mu \sin i'$, by substituting in the expression for D we have its minimum value. This value of D at the minimum, was formerly supposed to give the actual place of the rainbow, the ray SA being parallel to the direction of the shadow of the observer's head, and the ray DR entering the eye with the minimum deviation.

The argument for the visibility of the iris in this place was that at the minimum deviation the emergent rays were parallel to each other, and therefore gave a much brighter light than where they are dispersed at considerable inclinations. The existence of supernumerary rainbows seen often near the summit of the primary bow, were shewn by Dr Young to require interference for their explanation; and the author found the whole phenomena to belong to the interference near a caustic.

Referring to figure 79, Part I., let SEs be the direction of the shadow of the eye of the observer; r, v rain-drops from which the emergent rays enter the eye at the minimum deviation after *one* internal reflexion; r', v' drops from which they enter the eye at the minimum deviation after two internal reflexions. Then the radius of the primary rainbow was considered to be 180°—the minimum value of D after one reflexion; and that of the secondary rainbow was 180°—the minimum value of D after two internal reflexions.

The values of i and D depending on μ, are different for the different colours of the spectrum, and hence the brilliant separation of the colours.

The values of the deviations and the positions of the primary and secondary bows are discussed at page 83, Part I., and the directions of the asymptotes of the caustics for the first to the tenth iris in glass for $\mu = \frac{3}{2}$, are given at page 87.

ART. 49. PROP. *To trace the caustic after emergence when an incident pencil of parallel rays has been reflected a given* n *number of times within a refracting sphere or cylinder.*

Let O be the center of the circle in figure 40, which is the section of the sphere or cylinder, and SA the ray which falls perpendicularly upon the surface at A; and produce SA through O to T, the point of intersection with an emergent ray RP, or that ray produced.

Let $r = OA$ or OR be the radius of the circle, and $\rho = OP$ the distance of any point P in the emergent ray from O.

Then since the angle of emergence from a sphere equals the angle of incidence upon it, we have $\angle ORT = i$; let i' be the corresponding angle of refraction, and $\angle TOP = \theta$. We have $\angle OTP = 180^\circ - D$, the supplement of the deviation.

Then, as in the last proposition, we have

$$D = n\pi + 2i - 2 \, . \, (n+1) \, . \, i',$$

$$\sin i = \mu \sin i',$$

$$\text{and } \frac{di'}{di} = \frac{\cos i}{\mu \cos i'}.$$

Also, from the figure,

$$\frac{OP}{\sin i} = \frac{OR}{\sin OPR},$$

$$\text{or } \rho = r \, . \, \frac{\sin i}{\sin (POA - OTP)}$$

$$= r \, . \, \frac{\sin i}{\sin (D - \theta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Now when P is the intersection of two consecutive rays or a point in the caustic, we have ρ and θ constants whilst i and D vary; therefore differentiating (1) with these conditions, we have

$$0 = \sin(D-\theta)\cos i - \sin i \cos(D-\theta)\frac{dD}{di},$$

$$\text{or}\quad \tan(D-\theta) = \tan i \,.\, \frac{dD}{di} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Substituting the value of $\sin(D-\theta)$ from this in the value of ρ, we have

$$\rho = \frac{r \,.\, \sin i \sqrt{\left\{1 + \tan^2 i \left(\dfrac{dD}{di}\right)^2\right\}}}{\tan i \,.\, \dfrac{dD}{di}},$$

and from the first equations we have

$$\frac{dD}{di} = 2\left\{1 - (n+1)\frac{\cos i}{\mu \cos i'}\right\}.$$

Substituting this value and reducing, we have

$$\rho = r\sqrt{\left[\frac{\mu^2 \cos^2 i \,.\, \cos^2 i'}{4\{\mu \cos i' - (n+1)\cos i\}^2} + \sin^2 i\right]} \ldots\ldots\ldots (3);$$

putting the values of D and $\frac{dD}{di}$ in (2), we have

$$\tan\{n\pi + 2i - 2\,.\,(n+1)\,.\,i' - \theta\} = 2\tan i\left\{1 - (n+1)\,.\,\frac{\cos i}{\mu \cos i'}\right\} \quad (4);$$

and if i and i' were eliminated between the last two equations, and $\sin i = \mu \sin i'$, we should have for (3) the relation between ρ and θ required. Knowing the asymptote we easily however trace the form of the caustic from the equations (3) and (4).

Let $i = 0$ and therefore $i' = 0$, then $\theta = n\pi$, and

$$\rho = r \,.\, \frac{\mu}{2\{(n+1) - \mu\}},$$

where the ray SAT is a tangent to the caustic and which gives the commencing point of it.

Let $i = 90^\circ$, then $\rho = r$, and the caustic finally touches the circle whatever value μ may have.

To apply these results to the case of the primary rainbow, we have for orange-light $\mu = \frac{4}{3}$, and $n = 1$; therefore when $i = 0$, $\rho = r$.

Let *SBCDl* be the course of the ray which has the minimum deviation in figure 41, or let *ktDl* be the asymptote of the two branches *am* and *bn* of the caustic. Then the rays produced backwards which are incident between *A* and *B* are tangents to the branch *am*, and those which are incident beyond *B* from *A* are tangents to the branch *bn*. The luminiferous surface in one of its positions, as *ege'*, has the part from *e'* to the asymptote the involute of the branch *am*, and the remainder from the asymptote to the cusp *g* is the involute of the part of *bn* which is beyond *g* from *b*, whilst *ge* is the involute of the part *bg*.

The retardation δ between the two sheets of the luminiferous surface at any point depends upon the distance of that point from the asymptote and on the diameter of the rain-drop; and it may be calculated by tracing the paths of the two rays through the drop. This method, employed by Dr Young, enabled him to calculate that the second maximum intensity of the red would fall at the place of the first maximum of the violet, distant 2° from the first maximum of the red, if the rain-drop were $\frac{1}{76}$ inch in diameter. The author by assuming that the branches *eg* and *e'g* might be taken as the circles of curvature of those branches and to meet at *g*, obtained a similar result. This assumption shewed that the second maximum of the red would fall 1° 46′ from the first if the rain-drop were $\frac{1}{72}$ inch in diameter, These computations were both made on the supposition that there was brightness at the caustic where the rays have passed over equal spaces and $\delta = 0$; so that it is a desideratum in physical science that these subjects should receive a careful mathematical re-examination, since it is now certain that there is darkness at the caustic.

CHAPTER V.

ON THE INTERFERENCE OF POLARIZED LIGHT.

WHEN we speak of the properties of ordinary light as distinguished from those of polarized light, we only mean that the peculiar properties of polarized light do not come under consideration, because ordinary light may be considered as composed of light polarized in different planes. The properties of polarized light are effective to their full extent in this mixture nevertheless, and interference takes place, although it may not be evident from the light being polarized in different planes, or from the retardation of one beam behind another being so great that we may not be able to discern the particulars on account of their exceeding minuteness; and in both cases we may in this manner only perceive the aggregate effect. It is certain that all the phenomena of concurring beams are subject to the laws of interference, regular or irregular.

The phenomena of Newton's transmitted rings gave the interference formula

$$I = (l^2 + l'^2 + 2ll' \cos \theta)^{\frac{\nu\lambda}{2}};$$

and if L be the base of the interfering light so that

$$I = (L)^{\nu\lambda},$$

then L is represented by the diagonal of a parallelogram of which the sides represent l and l'; and the angle θ is expressed in terms of λ the interval, and δ the retardation, as follows:

$$\theta = \pi \left(1 \pm \frac{2\delta}{\lambda}\right),$$

and $\cos\theta = -\cos\left(2\,\pi\frac{\delta}{\lambda}\right)$,

so that $I = \left(l^2 + l'^2 - 2ll'\cos 2\pi\frac{\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}}$.

But where inversion of properties or anatropy exists,

$$\cos\theta = \cos\left(2\,\pi\frac{\delta}{\lambda}\right).$$

Now if we compound the two bases l and l' to find L by the above formula, we must use the inverse method to determine l and l' for a given value of L, in the resolution of the base into its components in given directions.

When the parallelogram is right-angled, or

$$\theta = \frac{\pi}{2} = \pi\left(1 \pm \frac{2\delta}{\lambda}\right),$$

we have $\delta = \frac{\lambda}{4}$.

So that if we resolve L into l and l' in directions at right angles it is equivalent to one of the components having a retardation $\pm\frac{\lambda}{4}$.

When a beam of polarized light whose base is L is incident upon the surface of a plate of a uniaxal crystal, and the plane of polarization makes the angle ϕ, as in figure 42, with the principal plane of the crystal; then taking the bases in the directions of the planes of the polarizations, and l being the base in the ordinary beam, l' that in the extraordinary beam, we have

$$l = L\cos\phi,$$
$$l' = L\sin\phi.$$

If I be the intensity of the incident beam, I_0 that of the ordinarily and I_e that of the extraordinarily refracted beam, we have

$$I = (L)^{\nu\lambda}$$
$$= (l^2 + l'^2)^{\frac{\nu\lambda}{2}},$$

$$\text{and} \quad I_0 = (l)^{\nu\lambda}$$
$$= (L \cos \phi)^{\nu\lambda},$$
$$I_e = (l')^{\nu\lambda}$$
$$= (L \sin \phi)^{\nu\lambda}.$$

These are the values of I_0 and I_e if they exist alone, but if they exist together we must recompound by the interference formula. It will be seen that these coincide with Malus's law only when $\nu\lambda = 2$.

ART. 50. When the bases l and l' are resolved again in rectangular directions as SO and ESE' in figure 43, we see that we have not L reproduced if SO coincides with the original plane of polarization, since l and l' furnish components in the perpendicular direction ESE'. The retardation which occurs on the second analyzation of the light requires an additional consideration. The retardation $\frac{\lambda}{4}$ between l and l' may be considered as made up of an acceleration for one as l, and a retardation for the other as l', proportional to the angles they make with L. Now l in So for an ordinary beam, and l' in Se for an extraordinary one, on being resolved again into SO an ordinary direction, and ESE' an extraordinary one, we may expect them to be accelerated and retarded in a like manner to what takes place on the first analyzation; that is, we shall have the component of l in SO accelerated and in SE' retarded; whilst the component of l' in SO is retarded and in SE is accelerated. A complete discussion must involve the methods of Chapter II.

In this manner the components in SO will have a retardation equal to $\frac{\lambda}{2}$. From this, in the interference formula we must take

$$\theta = 2\pi \frac{\delta}{\lambda},$$

$$\text{and} \quad I = \left(l^2 + l'^2 + 2ll' \cos 2\pi \frac{\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}}.$$

ART. 51. PROP. *To find the interference phenomena when light polarized in a given plane has been transmitted by a plate* *of a uniaxal crystal, cut perpendicular to the axis, and then analyzed.*

Let the beam of light entering the eye E have been polarized originally in the plane determined by the lines AB and SE in figure 44, taken as the plane of the paper, and before traversing the plate ab of the uniaxal crystal. After emergence from ab let it be analyzed in the plane determined by SE and CD, as a principal plane of a double refracting crystal, making an angle α with the plane of original polarization. Let cd represent any principal plane of the crystalline plate ab, making an angle ϕ with the plane of original polarization.

Let L be the base of the polarized light falling on ab; the base of the beam polarized in the principal plane $l_0 = L \cos \phi$; that polarized perpendicular to the principal plane $l_e = L \sin \phi$.

Putting l_{oo}, l_{eo}, the resolved parts of these in CD, and

$$\psi = \phi - \alpha,$$

we have

$$l_{oo} = L \cos \phi \cos \psi,$$

$$l_{eo} = L \sin \phi \sin \psi,$$

and compounding these again by the formula above, we have the intensity I as follows,

$$I = \left(l_{oo}^2 + l_{eo}^2 + 2 l_{oo} \,.\, l_{eo} \,.\, \cos 2\pi \frac{\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}}$$

$$= L^{\nu\lambda} \Big\{ \cos^2 \phi \cos^2 \psi + \sin^2 \phi \sin^2 \psi$$

$$+ 2 \sin \phi \cos \phi \sin \psi \cos \psi \left(1 - 2 \sin^2 \frac{\pi\delta}{\lambda}\right)\Big\}^{\frac{\nu\lambda}{2}}$$

$$= L^{\nu\lambda} \left\{ \cos^2 (\phi - \psi) - \sin 2\phi \,.\, \sin 2\psi \sin^2 \frac{\pi\delta}{\lambda} \right\}^{\frac{\nu\lambda}{2}}$$

$$= L^{\nu\lambda} \left\{ \cos^2 \alpha - \sin 2\phi \,.\, \sin 2\, (\phi - \alpha) \sin^2 \frac{\pi\delta}{\lambda} \right\}^{\frac{\nu\lambda}{2}} \text{............} (1).$$

First, to interpret this formula, let $\alpha = 0$, or let the plane of analyzation coincide with the plane of the original polarization, we have

$$I = L^{\nu\lambda}\left(1 - \sin^2 2\phi \sin^2 \frac{\pi\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}}.$$

When $\phi = 0$, or $\pm \frac{\pi}{2}$, or π, then the intensity is independent of δ, and $I = L^{\nu\lambda} =$ the intensity of the light falling on ab, and there is a bright cross as in figure 86 a, Part I.

When ϕ has any other given value there will be a series of maxima and minima depending on the value of δ.

When $\delta = 0$, or λ, 2λ, 3λ, &c.... $n\lambda$ and n any integer, then $\sin^2 \frac{\pi\delta}{\lambda} = 0$, and there will be maxima for all values of ϕ, and a brightness in the segments of rings between the arms of the cross, equal to that in them, as in fig. 86 a, Part I.

$$\text{When} \quad \delta = \frac{\lambda}{2}, \text{ or } \frac{3\lambda}{2}, \frac{5\lambda}{2}, \text{ \&c....} \frac{2n+1}{2}.\lambda,$$

$$\text{then} \quad \sin^2 \frac{\pi\delta}{\lambda} = 1;$$

and there will be minima where the brightness will be

$$I = L^{\nu\lambda}(1 - \sin^2 2\phi)^{\frac{\nu\lambda}{2}}.$$

But the segments of dark rings are not of uniform intensity like the bright ones. Where $\sin 2\phi = \pm 1$, $\phi = \pm \frac{\pi}{4}$, or $\pm \frac{3\pi}{4}$, there will be complete darkness, but not for other values of ϕ. This explains how the segments of the dark rings are only perfectly black at their middle parts between the arms of the bright cross. These particulars are easily seen in experiment, and as shewn in figure 86 a, Part I.

Intermediate values of δ give intermediate results.

Again, when $\alpha = \pm \frac{\pi}{2}$, we have $\cos \alpha = 0$, and

$$\sin 2(\phi - \alpha) = -\sin(180^\circ - 2\phi) = -\sin 2\phi;$$

$$\therefore\ I = L^{\nu\lambda}\left(\sin^2 2\phi\,.\,\sin^2\frac{\pi\delta}{\lambda}\right)^{\frac{\nu\lambda}{2}}.$$

When $\phi = 0$, or $\pm\frac{\pi}{2}$, or π, we have the intensity zero, or

$$I = 0,$$

and there is a dark cross, as in figure 86 *b*, Part I.

When ϕ has any other value, we have a series of maxima and minima, depending on the value of δ; when $\delta = 0$, or λ, 2λ, 3λ, &c.... $n\lambda$, and n any integer, we have the intensity $I = 0$, for all values of ϕ, and there are uniform dark segments of rings between the arms of the cross.

When $\delta = \frac{\lambda}{2}$, or $\frac{3\lambda}{2}$, $\frac{5\lambda}{2}$, ... $\frac{2n+1}{2}\lambda$, the intensity is

$$I = L^{\nu\lambda}\,(\sin^2 2\phi)^{\frac{\nu\lambda}{2}},$$

and there are bright segments of rings between the arms of the dark cross, but the intensity is not uniform, becoming only

$$I = L^{\nu\lambda} \quad \text{when} \quad \phi = \pm\frac{\pi}{4}, \quad \text{or} \ \pm\frac{3\pi}{4},$$

as seen readily in experiments, though not well represented in figure 86 *b*, Part I.

When α has any other values in (1), the intensity is uniform and independent of λ, becoming

$$I = L^{\nu\lambda}\,(\cos\alpha)^{\nu\lambda},$$

$$\text{when} \quad \sin 2\phi = 0, \quad \text{or} \ \sin 2\,(\phi - \alpha) = 0;$$

$$\text{or, when} \quad \phi = 0, \quad \text{or} \ \pm\frac{\pi}{2}, \quad \text{or} \ \pi;$$

$$\text{or, when} \quad (\phi - \alpha) = 0, \quad \text{or} \ \pm\frac{\pi}{2}, \quad \text{or} \ \pi.$$

That is, there is a double cross of the above intensity with the arms respectively parallel and perpendicular to the planes of original polarization and of analyzation.

For the values of ϕ between $\phi = 0$ and $\phi = \alpha$ we have

$$\sin 2\,(\phi - \alpha) = -\sin 2\,(\alpha - \phi),$$

and the expression for the intensity becomes in (1),

$$I = L^{\nu\lambda}\left\{\cos^2\alpha + \sin 2\phi \,.\, \sin 2(\alpha-\phi)\sin^2\frac{\pi\delta}{\lambda}\right\}^{\frac{\nu\lambda}{2}}.$$

For the values of ϕ between $\phi = \alpha$ and $\phi = \frac{\pi}{2}$, we have $\sin 2\phi$ and $\sin 2(\phi - \alpha)$, both positive, and

$$I = L^{\nu\lambda}\left\{\cos^2\alpha - \sin 2\phi \,.\, \sin 2(\phi-\alpha)\sin^2\frac{\pi\delta}{\lambda}\right\}^{\frac{\nu\lambda}{2}}.$$

Now for $\delta = 0$ these two values of I coincide, and we have the intensity of the crosses at the center as above; but when

$$\delta = \frac{\lambda}{2}, \quad \text{or } \frac{3\lambda}{2}, \text{ \&c.,}$$

we have a maximum in one case and a minimum in the other, or the segments of rings between the arms of the double cross have maxima in one sector coinciding in distance from the center with the minima in the adjacent sector.

The same will be found to hold for the other sectors.

Cor. In Art. (4), it was shewn that

$$\text{for negative crystals} \ldots \delta = \frac{b(a^2-b^2)}{2\mu a^2}\sin^2 i,$$

$$\text{for positive crystals} \ldots \delta = \frac{a(a^2-b^2)}{2\mu b^2}\sin^2 i.$$

Now the rays by which the rings are seen enter the eye obliquely and make an angle i with the axis; so that the angular diameters of the rings vary as $\sqrt{\delta}$, and are therefore similar to Newton's rings formed between spherical surfaces; becoming closer together as they are further from the center.

Art. 52. Prop. *On the nature of circularly polarized light.*

When a beam of light in the direction SO (fig. 45), perpendicular to the plane of the paper, is polarized in a plane whose intersection with the plane of the paper is OA, and this is decomposed into two beams having the same direction but polarized in planes determined by the lines SO, Oa, and Ob, inclined

$+45^\circ$ and -45° to the plane SOA, of which one is found, *on analyzation*, to be retarded $\frac{\lambda}{4}$ behind the other: the compound beam so constituted is called circularly polarized light. This name was first given from the properties being the same in all directions round O; but in the undulatory theory the name was applied to the circular vibrations of the particles of the hypothetical æther, which would then exist.

Let L be the base of the incident beam and the intensity

$$I = (L)^{\nu\lambda}.$$

Now when the plane of analyzation is SOB (figure 46), making the angle ϕ with SOA, we have for new bases l_1 and l_2, as follows:

$$\begin{aligned} l_1 &= L \cos 45^\circ . \cos (45^\circ - \phi) \\ &= \frac{L}{2} (\cos \phi + \sin \phi), \\ l_2 &= L \cos 45^\circ . \cos (45^\circ + \phi) \\ &= \frac{L}{2} (\cos \phi - \sin \phi); \end{aligned}$$

and if I' be the intensity of the beam analyzed in the plane SOB, we have in circularly polarized light the retardation $\delta = \frac{\lambda}{4}$; therefore

$$\begin{aligned} I' &= (l_1^2 + l_2^2)^{\frac{\nu\lambda}{2}} \\ &= \left[\frac{L^2}{4} \left\{(\cos \phi + \sin \phi)^2 + (\cos \phi - \sin \phi)^2\right\}\right]^{\frac{\nu\lambda}{2}} \\ &= \left(\frac{L}{\sqrt{2}}\right)^{\nu\lambda}; \end{aligned}$$

which is independent of ϕ, and the base in any plane of analyzation is $\frac{L}{\sqrt{2}}$. This property is most completely given to plane polarized light when it has been twice reflected as at A and B in figure 47, within a rhomb of glass of which the section is *abcd*, and each of the angles at a and c is 45°, in accordance with the result of Art. 20, the light entering the face *ab* perpendicularly and emerging perpendicularly from the face *cd* after two reflexions at A and B; the plane of incidence and reflexion being inclined 45° to the plane of polarization of the incident beam, and the angles of incidence at A and B being 45° from the results of Art. 20.

In several experiments with rhombs of plate and flint-glass, the retardation $\delta = \frac{\lambda}{4}$ appeared to be given by two reflexions immediately as total reflexion commenced, as stated at page 38, Part I., and with other rhombs the retardation appeared different, that is for higher angles of incidence at A and B. The experiment however must be tried with very perfectly annealed rhombs before a positive conclusion can be arrived at, and the theoretical result of Art. 20 confirmed.

Another method of M. Fresnel, which is very useful though not accurate, is to employ a very thin slip of mica to produce the effects required. That is, the thickness must be such as to give, on analyzation, the retardation

$$\delta = \frac{\lambda}{4}, \quad \text{or,}\ \delta = n\lambda \pm \frac{\lambda}{4},$$

and n is any integer, and the planes of polarization of the two rays must be inclined ± 45° to the plane of the original polarization of the light. Now mica being a biaxal crystal with the optic axes generally inclined at a large angle, the planes of polarization of the two rays bisect the acute and obtuse angles formed by planes through the optic axes, and the ray, as shown in figure 41, Part I.; and a lamina of proper thickness being attached to a tourmaline, with one of the planes of polarization of its rays inclined 45° to the plane of polarization of the light transmitted by the tourmaline, we have the conditions required for circular polarization, but only for such tint as the retardation is $\frac{\lambda}{4}$.

The circular polarization produced by the rhomb is called right-handed or left-handed, according as the plane of reflexion (in fig. 45) is inclined 45° to the right, or 45° to the left of the plane of original polarization SOA, and looking in the direction

SO, respectively; and we conclude that in total reflexions within transparent media when the angles of incidence at *A* and *B* are 45°, the light is analyzed into two parts, one polarized in the plane of incidence and the other perpendicular to it, and the former is retarded a space $\frac{\lambda}{8}$ at each reflexion.

When the inclination of the plane of reflexion to the plane of original polarization is any other angle than ± 45° the light is elliptically polarized, and the brightness is not constant for all values of ϕ; and when the inclination is 0°, or ± 90°, the light is plane polarized.

Art. 53. Prop. *To investigate the expression for the intensity of a beam of elliptically polarized light when analyzed in any given plane.*

In figure 48, let the letters refer to the similar points to those in fig. 46, but let Oa be now inclined at any angle β to OA; and one of the two beams being retarded $\frac{\lambda}{4}$ behind the other, the light is generally elliptically polarized, but becomes circularly or plane polarized in particular circumstances depending on the magnitude of β.

We have as in the last Article,

$$I' = (l_1^2 + l_2^2)^{\frac{\nu\lambda}{2}},$$

and now

$$l_1 = L \cos\beta \,.\, \cos(\beta - \phi),$$

$$l_2 = L \sin\beta \,.\, \sin(\beta - \phi);$$

$$\therefore\ I' = L^{\nu\lambda} \{\cos^2\beta \cos^2(\beta - \phi) + \sin^2\beta \,.\, \sin^2(\beta - \phi)\}^{\frac{\nu\lambda}{2}}.$$

When $\beta = 0$, or 90°,

$$I' = L^{\nu\lambda} \,.\, (\cos\phi)^{\nu\lambda},$$

and

$$\phi = 0, \quad \text{then} \quad I' = L^{\nu\lambda},$$

$$\phi = 90^\circ, \quad \text{then} \quad I' = 0;$$

or, the light is plane polarized as found by experiment.

When $\beta = 45^\circ$, $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$;

$$\therefore\ I' = L^{\nu\lambda}\left[\frac{1}{2}\left\{\cos^2(\beta-\phi)+\sin^2(\beta-\phi)\right\}\right]^{\frac{\nu\lambda}{2}}$$

$$= \left(\frac{L}{\sqrt{2}}\right)^{\frac{\nu\lambda}{2}},$$

as found directly for circularly polarized light.

ART. 54. PROP. *To find the phenomena when circularly polarized light falls upon a plate of a uniaxal crystal, cut perpendicularly to the optic axis, and the transmitted light has been analyzed.*

Let *SAe* (in fig. 49) be the direction of the ray which passes along the axis of the uniaxal plate *ab*, and enters the eye at *e*.

Let *SAB* be the plane of the original polarization of the incident light, which has been brought to the state of circular polarization as described in Article 52, and then consisting of two equal portions polarized in the planes *SAE* and *SAF* inclined $+45^\circ$ and -45° to the plane *SAB*, and the latter retarded $\frac{\lambda}{4}$ behind the former, or it is right-handed circular polarization.

Let a principal plane *Scd* of the crystalline plate *ab* make an angle ϕ with the plane *SAB*, and let the plane of analyzation *SCD* make an angle α with *SAB*.

Now if L be the base of the incident light originally polarized in the plane *SAB*, we have the bases of that polarized in the principal plane *Scd* equal to $\frac{L}{\sqrt{2}}$, as shown in Art. 52, as well as that polarized in the plane at right angles to it, as *Scf*.

The ordinary ray being retarded δ behind the extraordinary ray, we have, after analyzation in the plane *SCD*, the final retardation $\delta' = \delta - \frac{\lambda}{4}$; and the component bases are

$$l = \frac{L}{\sqrt{2}}\cos(\phi-\alpha),$$

$$l' = \frac{L}{\sqrt{2}}\sin(\phi-\alpha),$$

and the base L' squared, of the interfering light entering the eye, is

$$
\begin{aligned}
L'^2 &= l^2 + l'^2 + 2ll' \cos 2\pi \frac{\delta'}{\lambda} \\
&= \frac{L^2}{2}\left\{\cos^2(\phi-\alpha) + \sin^2(\phi-\alpha) + 2\sin(\phi-\alpha)\cos(\phi-\alpha)\cos 2\pi\frac{\delta'}{\lambda}\right\} \\
&= \frac{L^2}{2}\left\{1 + \sin 2\,(\phi-\alpha)\cos 2\pi\frac{\delta'}{\lambda}\right\}.
\end{aligned}
$$

Let $(\phi-\alpha) = \psi$ the angle between the plane of analyzation and the principal plane of the plate ab; then if I be the intensity of the light entering the eye, we have

$$I = \left(\frac{L}{\sqrt{2}}\right)^{\nu\lambda}\left(1 + \sin 2\psi \cos 2\pi\frac{\delta'}{\lambda}\right)^{\frac{\nu\lambda}{2}};$$

which is independent of α and ϕ, and depends only upon their difference ψ, so that on turning round the analyzer we have the same appearances.

When $\psi = 0$, or $\pm\frac{\pi}{2}$, or π, we have

$$I = \left(\frac{L}{\sqrt{2}}\right)^{\nu\lambda};$$

which being independent of δ', we have a cross partially bright, as in figure 87, Part I., with its arms parallel and perpendicular to the plane of analyzation.

When $\psi = +\frac{\pi}{4}$, or $\frac{5\pi}{4}$, we have $\sin 2\psi = 1$, and the maxima intensities occur when $\cos 2\pi\frac{\delta'}{\lambda} = 1$, or when $\delta' = 0$, or λ, 2λ, &c. ... $n\lambda$, and n any integer, or when

$$\delta = \delta' + \frac{\lambda}{4} = \frac{\lambda}{4}, \text{ or } \frac{5\lambda}{4}, \frac{9\lambda}{4}, \frac{13\lambda}{4}, \text{ \&c.};$$

and the minima are black when $\cos 2\pi\frac{\delta'}{\lambda} = -1$, or when

$$\delta' = \frac{\lambda}{2}, \text{ or } \frac{3\lambda}{2}, \frac{5\lambda}{2}, \text{ \&c.} \ldots \frac{2n+1}{2}\lambda,$$

$$\text{or } \delta = \delta' + \frac{\lambda}{4} = \frac{3\lambda}{4}, \text{ or } \frac{7\lambda}{4}, \frac{11\lambda}{4}, \text{ \&c.}$$

Again, when $\psi = -\frac{\pi}{4}$, or $\frac{3\pi}{4}$, then $\sin 2\psi = -1$, and the maxima occur for the values of δ which gave the minima in the other sectors, and the minima for those which gave the maxima in the former case.

The segments of the rings in the alternate sectors hence differ by a retardation of half an interval; or the distances from the center of the bright rings in one sector equal those of the dark ones in the neighbouring sectors, as seen in the experiments, and shown in figure 87, Part I.

ART. 55. PROP. *To find the phenomena when the light incident upon a plate of a uniaxal crystal is circularly polarized, and the light emergent from it is circularly analyzed.*

Let *SCD* (in fig. 50) be the plane of final analyzation making an angle $\psi = \alpha - \phi$ with *Scd*, any principal plane of the plate of uniaxal crystal *ab*. Then the planes inclined ± 45° to *SCD* will be those into which the first analyzations must be made of the bases equal to $\frac{L}{\sqrt{2}}$, emergent from the plate *ab*, polarized in the principal plane and perpendicular to it; and we have from the principal plane the components

$$\frac{L}{\sqrt{2}} \cos (45^\circ + \psi) = \frac{L}{2} (\cos \psi - \sin \psi),$$

$$\frac{L}{\sqrt{2}} \cos (45^\circ - \psi) = \frac{L}{2} (\cos \psi + \sin \psi).$$

These analyzed finally in the plane *SCD*, by multiplying by

$$\cos (\pm 45^\circ) = \frac{1}{\sqrt{2}},$$

and then compounded with the retardation $\frac{\lambda}{4}$, give a base L_0 as follows:

$$L_0^2 = \frac{L^2}{8}\{(\cos\psi - \sin\psi)^2 + (\cos\psi + \sin\psi)^2\}$$

$$= \frac{L^2}{4}.$$

In the same way we have from the emergent light polarized perpendicular to the principal plane of the crystal the components

$$\frac{L}{\sqrt{2}}\cos(90^\circ + 45^\circ + \psi) = -\frac{L}{\sqrt{2}}\sin(45^\circ + \psi)$$

$$= -\frac{L}{2}(\cos\psi + \sin\psi),$$

$$\frac{L}{\sqrt{2}}\cos(90^\circ - 45^\circ + \psi) = \frac{L}{\sqrt{2}}\sin(45^\circ - \psi)$$

$$= \frac{L}{2}(\cos\psi - \sin\psi).$$

These compounded with the retardation $\frac{\lambda}{4}$ give for the base L_e as follows, after analyzation, by multiplying by cos (± 45°),

$$L_e^2 = \frac{L^2}{8}\{(\cos\psi + \sin\psi)^2 + (\cos\psi - \sin\psi)^2\}$$

$$= \frac{L^2}{4}.$$

Finally, L_0 and L_e compounded with the retardation δ'' give

$$L'^2 = L_0^2 + L_e^2 + 2L_0 . L_e \cos 2\pi\frac{\delta''}{\lambda}$$

$$= \frac{L^2}{4}\left(2 + 2\cos 2\pi\frac{\delta''}{\lambda}\right)$$

$$= \frac{L^2}{2}\left(1 + \cos 2\pi\frac{\delta''}{\lambda}\right);$$

or the intensity does not change with ψ, and there are no crosses bright or dark, but complete rings, the maxima occurring where

$$\cos 2\pi\frac{\delta''}{\lambda} = +1,$$

and the black rings where

$$\cos 2\pi\frac{\delta''}{\lambda} = -1;$$

the intensities of the bright rings being $I' = (L)^{\nu\lambda}$, and those of the dark ones zero, or $I' = 0$.

Art. 56. Prop. *To find the phenomena when a plate of quartz crystal cut perpendicularly to the axis is placed between the polarizing and analyzing plates.*

In figure 51, let *SAaCe* be a ray entering the eye *e*, originally polarized in the plane *SAB*.

Let *Sab* be a principal plane of the plate of quartz, making an angle ϕ with the plane of original polarization *SAB*, and let *SCD* be the plane of analyzation, making the angle α with *SAB*.

From Art. 53 the base of an elliptically polarized beam after analyzation is expressed by the formula

$$l^2 = L^2(\cos^2\beta\cos^2\psi + \sin^2\beta\sin^2\psi),$$

where ψ is the angle in fig. 48 between the chief plane of polarization of the elliptically polarized light and the plane of final analyzation *SCD*; but $\cos\beta$ and $\sin\beta$ are now to be considered coefficients only, which determine the degree of ellipticity of the two beams into which the light is refracted by the quartz plate, according to the experiments of M. Biot as interpreted by M. Fresnel, which are circularly polarized when they pass along the axis of the quartz plate, and elliptically polarized near it, but plane polarized when they pass at right angles to it; the two beams traversing the axis of the crystal with slightly different velocities; and the light becoming plane polarized to the senses at an inclination of 20 degrees to the axis, as found by M. Jamin.

In the formula above we have $\beta = 45°$ at the axis, and $\beta = 0$ at right angles to it, and β also becoming sensibly zero at an inclination of 20 degrees to the axis. If L be the base of the light incident upon the quartz plate, polarized in the plane *SAB*, we must put $L\cos\phi$ for that of the ordinary beam, and $L\sin\phi$

for that of the extraordinary beam when the transmitted light is sensibly plane polarized; but $\cos\phi$ and $\sin\phi$ become each $\frac{1}{\sqrt{2}}$ at the axis, or $\phi = 45^\circ$, and ϕ changes with β near to it. Thus if l_0 and l_e are the bases of the two components after analyzation, we have

$$l_0^2 = L^2 \cos^2\phi\,(\cos^2\beta\cos^2\psi + \sin^2\beta\sin^2\psi),$$
$$l_e^2 = L^2 \sin^2\phi\,(\cos^2\beta\sin^2\psi + \sin^2\beta\cos^2\psi)\,;$$

the latter being formed for the plane *Sac* as the former was for the plane *Sab*, at right angles to it.

These are to be compounded by the formula, with the retardation δ', thus:

$$L'^2 = l_0^2 + l_e^2 + 2l_0 . l_e \cos 2\pi\frac{\delta'}{\lambda}\,;$$

or substituting the values above, we have

$${}^2 = L^2\Big[\cos^2\phi(\cos^2\beta\cos^2\psi+\sin^2\beta\sin^2\psi)+\sin^2\phi(\cos^2\beta\sin^2\psi+\sin^2\beta\cos^2\psi)$$
$$-2\sin\phi\cos\phi\sqrt{\{(\cos^2\beta\cos^2\psi+\sin^2\beta\sin^2\psi)(\cos^2\beta\sin^2\psi+\sin^2\beta\cos^2\psi)\}}\cos 2\pi\frac{\delta'}{\lambda}\Big],$$

and the intensity I is

$$I = (L')^{\nu\lambda}.$$

Now at the axis we have $\beta = 45^\circ$, and $\cos\beta = \frac{1}{\sqrt{2}} = \sin\beta$, and

$$L'^2 = L^2\left[\frac{\cos^2\phi}{2}(\cos^2\psi + \sin^2\psi) + \frac{\sin^2\phi}{2}(\cos^2\psi + \sin^2\psi)\right.$$
$$\left. + 2\sin\phi\cos\phi\sqrt{\frac{\{(\sin^2\psi + \cos^2\psi)^2\}}{4}}\cos 2\pi\frac{\delta'}{\lambda}\right]$$
$$= \frac{L^2}{2}\left(1 + \sin 2\phi\cos 2\pi\frac{\delta'}{\lambda}\right);$$

and since $\phi = 45^\circ$ also, we have

$$L'^2 = \frac{L^2}{2}\left(1 + \cos 2\pi\frac{\delta'}{\lambda}\right).$$

Since ψ has disappeared, the intensity is the same all around the axis, and near to it, whilst $\beta = 45^0$ nearly, or the light can be considered as consisting of two circularly polarized beams. The intensity depends upon $\cos 2\pi \frac{\delta'}{\lambda}$, and δ' is a function of α, the angle between the planes of polarization and analyzation, the phenomena recurring with every 180^0 that α is increased, so that we have

$$I = (L')^{\nu\lambda} = \left(\frac{L}{\sqrt{2}}\right)^{\nu\lambda} \left(1 + \cos 2\pi \frac{\delta'}{\lambda}\right)^{\frac{\nu\lambda}{2}}.$$

That δ' is a function of α we may see in the following manner. In fig. 52 let SE be a ray which has traversed the quartz plate AB, the transmitted light being circularly polarized with ϕ taken 45^0. Let the lines drawn perpendicularly to SE through the points o, o', e, e' represent the sections of the luminiferous surfaces constituting the circularly polarized light, in contemporaneous positions with the distances oo' and ee', each equal to $\frac{\lambda}{4}$, and of which the polarizations are indicated by the small letters ⊢, parallel and perpendicular to the plane of the figure. Now if the crystal were an ordinary uniaxal one, the retardation would be the distance oe, but in the circularly polarized light, putting $\alpha' = \alpha - \frac{\pi}{4}$, when the plane of analyzation coincides with the plane of the figure, or $\alpha' = 0$, the light furnishes no components from o' and e, and the retardation is $oe' = oe + \frac{\lambda}{4}$; and when perpendicular to it, or $\alpha' = \frac{\pi}{2}$, the light furnishes no components from o and e', and the retardation is $o'e = oe - \frac{\lambda}{4}$. For intermediate positions the retardation will be intermediate and involve a function of α'. We may thus assume an expression fulfilling the above conditions in the following form:

$$\delta' = oe + \frac{\lambda}{4} - \frac{\lambda}{2} \cdot \frac{\alpha'}{\frac{\pi}{2}} = oe + \frac{\lambda}{2} - \frac{\alpha' + \frac{\pi}{4}}{\pi} . \lambda;$$

or, putting $\delta = oe$, and $\alpha = \alpha' + \frac{\pi}{4}$, we have

$$\delta' = \delta + \frac{\lambda}{2} - \frac{\alpha\lambda}{\pi}.$$

Now when $\alpha = \frac{\pi}{2}$ in fig. 51, before the crystal was interposed there was darkness; and $I = 0$ again when the crystal being interposed we have $\cos 2\pi \frac{\delta'}{\lambda} = -1$. Then $\alpha = \alpha' + \frac{\pi}{4}$, for this value of δ' will be the angle through which the plane of analyzation must be turned to reproduce the dark field;

or $\delta' = \frac{2n+1}{2}\lambda = \delta + \frac{\lambda}{2} - \frac{\alpha\lambda}{\pi}$, where $n = 0$, or any integer;

$$\therefore \; n\lambda = \delta - \frac{\alpha\lambda}{\pi};$$

and when $n = 0$ for thin plates we have

$$\frac{\alpha}{\pi} = \frac{\delta_0}{\lambda}.$$

For a thickness one millimetre or ·03937 inch and orange-light, M. Jamin found $\frac{\delta_0}{\lambda} = {\cdot}12$, and hence

$$\alpha = {\cdot}12\pi = {\cdot}12 \times 180^{\circ} = 21^{\circ}{\cdot}6,$$

which agrees with the experiments of M. Biot.

For a series of thicknesses and values of δ in arithmetic progression, we shall have the corresponding values of α in arithmetic progression; according to M. Biot's law of rotatory polarization.

If T be the thickness of the quartz plate, a a given linear constant equal to δ when $T = \lambda$, and $\delta = \frac{Ta}{\lambda}$, we have $\alpha = \frac{\pi Ta}{\lambda^2}$ varying as $\frac{1}{\lambda^2}$ for different colours with the same plate.

When $\beta = 0$, we have

$$L'^2 = L^2 \Big(\cos^2\phi \cos^2\psi + \sin^2\phi \sin^2\psi$$
$$+ 2 \sin\phi \cos\phi \sin\psi \cos\psi \cos 2\pi \frac{\delta'}{\lambda}\Big)$$
$$= L^2 \left\{(\cos\phi \cos\psi + \sin\phi \sin\psi)^2 - \sin 2\phi \sin 2\psi \sin^2 \frac{\pi\delta'}{\lambda}\right\}$$
$$= L^2 \left\{\cos^2(\phi \sim \psi) - \sin 2\phi \sin 2\psi \sin^2 \frac{\pi\delta'}{\lambda}\right\}$$
$$= L^2 \left(\cos^2 \alpha - \sin 2\phi \sin 2\psi \sin^2 \frac{\pi\delta'}{\lambda}\right),$$

as found for a uniaxal crystal in Art. 51.

At different inclinations to the axis of the crystal, it was formerly supposed that the retardation of the extraordinary ray behind the ordinary one varied as a constant for the axis, plus the square of the inclination to the axis, analogous to the case of calc-spar and other uniaxal crystals. The progression of the rings are however evidently very different from this near the axis, and M. Jamin* has found that the formulæ of M. Cauchy represent excellently the phenomena. The following are from his results. Let δ_0 be the difference of paths for the axis and a thickness of the plate one millimetre or ·03937 inch or $\delta_0 = {\cdot}12\lambda$, i' = angle of refraction within the plate = inclination of the ray to the optic axis, μ the refractive index of the ordinary ray, when incident at right angles to the axis of the crystal, μ' that of the extraordinary ray; then $\mu' - \mu = {\cdot}00905$ from the experiments.

The thickness of the plate being a millimetre, δ the retardation at an inclination i', by M. Cauchy's formulæ we have

$$p = \frac{\mu}{\mu'} \cdot \frac{\mu' - \mu}{\lambda},$$
$$\delta^2 = p^2 \sin^4 i' + \delta_0^2 \cos^4 i';$$

also the ratio of the component bases in the elliptically polarized light he found to be expressed by the formula,

---

* *Annales de Chimie et de Physique*, Tom. xxx. p. 56.

$$\text{the ratio} = k = \pm\left\{\frac{p}{\delta_0}\tan^2 i'' \pm \sqrt{\left(1 + \frac{p^2}{\delta_0^2}\tan^4 i''\right)}\right\}.$$

In the experiments p was found

$= 16{\cdot}034$, for $\lambda = {\cdot}000561^{mm} = {\cdot}000022086$ inch,

for the orange-light of the spirit-lamp with salted wick.

There is, however, a fatal objection to the generality of these formulæ, for they are not homogeneous.

The quantity p for a thickness one millimetre and λ measured in decimals of a millimetre, is a ratio and numerical; but δ and δ_0 are linear, and their values depend on the unit of linear measure which is used, but of which p is independent. A similar objection lies against the formula for k. The value of λ employed is also considerably different from Fraunhofer's value for the fixed line D, which is generally allowed to correspond with the orange-light of the spirit-lamp with salted wick. It must therefore be considered that the accordance of M. Jamin's experimental results with the formulæ is accidental, and that the true general formulæ are yet to be discovered.

The formula given by M. Cauchy for the retardation δ in the *Comptes Rendus*, Tome XXX. p. 99, is

$$\delta^2 = \delta_0^{\,2}\cos^4\tau' + \epsilon^2\sin^4\tau',$$

and c being the thickness of a plate of quartz cut perpendicular to the axis; also $\frac{\epsilon}{c}$ being the difference between the indices of ordinary and extraordinary refraction, with τ' the angle of refraction of the ordinary ray.

This formula is homogeneous, but must have been subsequently changed.

ART. 57. PROP. *To find the phenomena when circularly polarized light is incident upon a plate of quartz crystal cut perpendicularly to the axis, and the emergent light has been analyzed.*

In figure 53 the letters referring to the same parts as in the last proposition, the incident light upon the crystal now consists of two equal portions polarized in planes inclined ± 45° to the plane of original polarization *SAB*, with a retardation $\frac{\lambda}{4}$ between them.

Therefore putting $(\phi - 45^\circ)$ and $(\phi + 45^\circ)$ for ϕ in the expression of the last proposition, we have

$$l_0^2 = L^2 \cos^2 (\phi - 45^\circ) (\cos^2 \beta \cos^2 \psi + \sin^2 \beta \sin^2 \psi) \ldots\ldots (1),$$

$$l_0'^2 = L^2 \cos^2 (\phi + 45^\circ) (\cos^2 \beta \cos^2 \psi + \sin^2 \beta \sin^2 \psi) \ldots\ldots (2),$$

$$l_e^2 = L^2 \sin^2 (\phi - 45^\circ) (\cos^2 \beta \sin^2 \psi + \sin^2 \beta \cos^2 \psi) \ldots\ldots (3),$$

$$l_e'^2 = L^2 \sin^2 (\phi + 45^\circ) (\cos^2 \beta \sin^2 \psi + \sin^2 \beta \cos^2 \psi) \ldots\ldots (4).$$

Now (1) and (2) have the retardation $\frac{\lambda}{4}$, as also (3) and (4), so that if L' is the base of the finally analyzed light from compounding (1) and (2), we have

$$\begin{aligned} L'^2 &= L^2 [(\cos^2\beta \cos^2\psi + \sin^2\beta \sin^2\psi) \{\cos^2 (\phi - 45^\circ) + \cos^2 (\phi + 45^\circ)\}] \\ &= L^2 [(\cos^2\beta\cos^2\psi + \sin^2\beta\sin^2\psi) \{\tfrac{1}{2}(\cos\phi + \sin\phi)^2 + \tfrac{1}{2}(\cos\phi - \sin\phi)^2\}] \\ &= L^2 (\cos^2\beta \cos^2\psi + \sin^2\beta \sin^2\psi). \end{aligned}$$

Similarly, if L'' is the base from compounding (3) and (4), we have

$$\begin{aligned} L''^2 &= L^2 [(\cos^2\beta \sin^2\psi + \sin^2\beta \cos^2\psi) \{\sin^2 (\phi - 45^\circ) + \sin^2 (\phi + 45^\circ)\}] \\ &= L^2 (\cos^2\beta \sin^2\psi + \sin^2\beta \cos^2\psi), \end{aligned}$$

and L'' has a retardation δ' behind L'; therefore if **L** is the base of the interfering light entering the eye, we have

$$\begin{aligned} \mathbf{L}^2 &= L'^2 + L''^2 + 2L'L'' \cos 2\pi \frac{\delta'}{\lambda} \\ &= L^2 \Big[1 + 2\sqrt{\{(\cos^2\beta\cos^2\psi + \sin^2\beta\sin^2\psi)(\cos^2\beta\sin^2\psi + \sin^2\beta\cos^2\psi)\}} \cos 2\pi \frac{\delta}{} \\ &= L^2 \left\{ 1 + \sqrt{(1 - \cos^2 2\beta \cos^2 2\psi)} \, . \cos 2\pi \frac{\delta'}{\lambda} \right\}. \end{aligned}$$

Now at the axis of the crystal $\beta = 45^\circ$, and $\cos\beta = \sin\beta = \frac{1}{\sqrt{2}}$;

and substituting these values, we have

$$\mathrm{L}^2 = L^2\left(1 + \cos 2\pi\frac{\delta'}{\lambda}\right),$$

which is independent of ψ; and therefore at the axis and near to it the light is uniform on all sides. But δ' in Art. 56, having the expression

$$\delta' = \delta + \frac{\lambda}{2} - \frac{\alpha\lambda}{\pi},$$

the value of L depends upon the value of α, the angle between the planes of polarization and final analyzation, as well as upon δ or the thickness of the plate; and the values of $\cos 2\pi\frac{\delta'}{\lambda}$ recur as α is increased by π in value, and change sign as α increases every $\frac{\pi}{2}$ in value. These are easily seen in the experiment; for if in place of the arrangement which produces the appearance of fig. 90, Part I., we turn the plane of analyzation round 90°, we find brightness in place of darkness in that figure, and darkness in place of brightness.

Again as i' increases from 0 at the axis, β rapidly diminishes from 45° at the axis, until it becomes sensibly zero when i' is 20°, and $\cos 2\beta$ increases from zero at the axis to unity when $i' =$ or $> 20°$, and then

$$\mathrm{L}^2 = L^2\left\{1 + \sqrt{(1 - \cos^2 2\psi)}\cos 2\pi\frac{\delta'}{\lambda}\right\}$$

$$= L^2\left\{1 + \sin 2\psi \cos 2\pi\frac{\delta'}{\lambda}\right\};$$

and there are now brighter and darker spaces depending on ψ and δ'. Taking for different values of i' whilst α is constant, until the true expression is found, the approximate expression

$$\delta'^2 = \delta_0^2 \cos^4 i' + \epsilon^2 \sin^4 i',$$

where ϵ^2 is still an unknown function of the thickness of the plate; then if $\sin 2\psi \cos 2\pi\frac{\delta'}{\lambda}$ were constant, we should have the relation of i' and ψ in the spiral curves at a distance from the axis.

Near the axis however the spiral curves, bright or dark, would be found by putting L^2 constant, or

$$\sqrt{(1 - \cos^2 2\beta \cos^2 2\psi)} \,.\, \cos 2\pi \frac{\delta'}{\lambda} = \text{constant},$$

in which we might take $\tan\beta = \left(\frac{1 - \tan i'}{1 + \tan i'}\right)^4$ as satisfying the conditions of the rapidly diminishing value of β from 45° when $i' = 0$, to β very small when $i' = 20°$; and then the relations of ψ and i' would furnish the equations of the spirals.

ART. 58. PROP. *To find the form of the interference curves when a plate of a biaxal crystal cut perpendicularly to the line bisecting the angle between the optic axes is placed between the polarizing and analyzing plates.*

The incident light being polarized in the plane *SAB*, let the ray *SOe* traverse the plate of the biaxal crystal along the line which bisects the angle between the optic axes a_1 and a_2 in the figures 54 and 55, supposed upon the second surface of the plate which is perpendicular to *SOe*, and *P* the point where any ray emerges. If we join a_1, a_2 and *P* as in the lower figure, and draw *Pg* bisecting the angle a_1Pa_2, then the plane of polarization of one of the rays passes through *Pg*, as stated at page 40, Part I., and the plane of polarization of the other is at right angles to it.

The letters in the figure being in a plane at right angles to the line *SAO*, let ϕ = angle between the plane of polarization of the first ray and that of the original polarization, or between *Pg* and *OH*; let γ = angle between the plane of the optic axes and the plane of original polarization, or between a_1a_2 and *OH*. Join a_1P, a_2P, and *OP*; put $OP = \rho$ = radius vector, and angle $POa_1 = \theta$; also put $a_1O = a_2O = c$ = half the distance of the optic axes.

Now if δ = retardation of one ray behind the other, as they emerge from the plate at *P*, we have, from Art. 14,

$$a_1P \times a_2P = \frac{\delta}{k},$$

where k is a given constant, and δ is constant for a given interference curve; which gives

$$a_1P \times a_2P = \text{constant},$$

the property of the lemniscate curves.

To find the equation of the curve for a given value of δ in terms of ρ and θ, we have from the triangle a_1PO,

$$a_1P^2 = \rho^2 + c^2 - 2\rho c \cos\theta;$$

and from the triangle a_2PO,

$$a_2P^2 = \rho^2 + c^2 + 2\rho c \cos\theta;$$

$$\therefore\ a_1P^2 \times a_2P^2 = \frac{\delta^2}{k^2}$$

$$= (\rho^2 + c^2)^2 - 4\rho^2c^2\cos^2\theta;$$

which gives

$$\rho^2 = c^2(2\cos^2\theta - 1) \pm \sqrt{\left[c^4\{(2\cos^2\theta - 1)^2 - 1\} + \frac{\delta^2}{k^2}\right]},$$

where the + sign is to be employed.

To find the curve with δ constant, which passes through O, we have at that point

$$a_1P = a_2P = c, \quad \text{and } \frac{\delta^2}{k^2} = c^4;$$

$$\therefore\ \rho^2 = 2c^2(2\cos^2\theta - 1)$$

$$= 2c^2\cos 2\theta,$$

which is the equation of the lemniscate of Bernoulli, as in fig. 56, $OPAA'$, and as seen in the experiments.

For points near the optic axis a_1 putting $a_2P = 2c$ nearly, then $a_1P =$ constant nearly when δ is constant, and the curves are ovals nearly circles. Also since $\frac{\delta}{k}$ varies as a_1P nearly, therefore giving δ a succession of values increasing by λ, we have a series of curves at nearly equal distances, as shewn in fig. 94, Part I. For points at a great distance from the optic axes when they are near together, we have $a_1P = a_2P$ nearly, and $a_1P^2 =$ constant, nearly, for δ constant; or the interference curves are again nearly circles.

These figures are seen the best with homogeneous light, and by using circular polarization and circular analyzation, so that the brushes disappear.

ART. 59. PROP. *To find the forms of the brushes when a plate of a biaxal crystal is placed between the polarizing and analyzing plates.*

Taking the figures of the last proposition, produce the line $a_1 a_2$ to b, and let the angle $Pa_1b = a_1$, angle $Pa_2b = a_2$.

We have angle $Pgb = \phi - \gamma$, and since the line Pg bisects the angle a_1Pa_2, we have

$$\text{angle } a_1Pg = \text{angle } a_2Pg,$$

$$\text{or} \quad a_1 - (\phi - \gamma) = (\phi - \gamma) - a_2,$$

$$\text{or} \quad a_1 + a_2 = 2(\phi - \gamma);$$

$$\text{put} \quad 2\eta = 90^\circ - 2(\phi - \gamma), \quad \text{or } \eta = 45^\circ - (\phi - \gamma),$$

$$\text{and} \quad \tan(a_1 + a_2) = \tan 2(\phi - \gamma) = \cot 2\eta;$$

$$\text{but} \quad \tan a_1 = \frac{\rho \sin\theta}{\rho\cos\theta - c}, \quad \text{and} \quad \tan a_2 = \frac{\rho\sin\theta}{\rho\cos\theta + c};$$

$$\therefore \tan(a_1 + a_2) = \cot 2\eta$$

$$= \frac{\tan a_1 + \tan a_2}{1 - \tan a_1 . \tan a_2}$$

$$= \frac{2\rho^2 \sin\theta\cos\theta}{\rho^2(\cos^2\theta - \sin^2\theta) - c^2},$$

$$\text{or} \quad \cos 2\eta\,(\rho^2\cos 2\theta - c^2) = \rho^2 \sin 2\theta . \sin 2\eta;$$

$$\therefore \rho^2 = \frac{c^2\cos 2\eta}{\cos 2(\theta + \eta)} = \frac{c^2\cos 2\eta}{2\cos^2(\theta + \eta) - 1}.$$

Comparing with $\rho^2 = \dfrac{a^2(e^2 - 1)}{e^2\cos^2\theta' - 1}$ the polar equation of an hyperbola referred to the center, we see that whilst η is constant, we have $e^2 = 2$, and $a^2 = c^2\cos 2\eta$, and the equation is that of a rectangular hyperbola.

Now since the rays are polarized at right angles to each other, on emerging from the biaxal plate, we have the same relation as in uniaxal crystals until we come to interpret the expressions; and therefore as in Art. 51, we have the intensity I as follows:

$$I = L'^{\nu\lambda},$$

$$\text{and} \qquad L'^2 = L^2\left(\cos^2\alpha - \sin 2\phi \sin 2\psi \sin^2 \pi\frac{\delta}{\lambda}\right).$$

In the brushes the intensity is independent of δ, and therefore

$$\sin 2\phi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\sin 2\psi = \sin 2(\phi - \alpha) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and from (1) ... $\phi = 0$, or $\phi = 90°$,

from (2) ... $\phi = \alpha$, or $\phi = 90° + \alpha$.

Taking the second values of ϕ from these, we have

$$\begin{aligned} \eta &= 45° - (\phi - \gamma) = 45° - 90° + \gamma \\ &= \gamma - 45°, \end{aligned}$$

$$\text{and} \qquad \begin{aligned} \eta &= 45° - (90° + \alpha - \gamma) \\ &= \gamma - \alpha - 45°. \end{aligned}$$

If we put θ' for $\theta + \eta$, or put $\rho^2 = \dfrac{c^2 \cos 2\eta}{2\cos^2\theta' - 1}$, we have

$$\theta' = \theta + \gamma - 45° \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\theta' = \theta + \gamma - \alpha - 45° \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

These shew that there are generally two hyperbolic brushes which pass through each optic axis, since $\theta = 0$ gives $\rho^2 = c^2$, and their transverse axes are inclined 45° to the planes of original polarization and analyzation; or their asymptotes are parallel and perpendicular to those planes respectively.

When $\alpha = 0$, or 90°, the two hyperbolas coincide, whatever γ may be, and the intensity $I = (L \cos \alpha)^{\nu\lambda}$, being $L^{\nu\lambda}$ in the former case, and 0 in the latter.

When $\gamma = 0$, or 90°, in (3), or $\gamma = \alpha$, or $\alpha + 90^\circ$, in (4),

we have $\eta = \pm 45^\circ$, and $\cos 2\eta = 0$,

but $$\rho^2 = \frac{c^2 \cos 2\eta}{\cos 2(\theta + \eta)};$$

and we must have $\theta = 0$, or 90°, or the hyperbolas are reduced to their asymptotes, giving a bright or dark cross through the center O.

Art. 60. The discussion of the equation

$$L'^2 = L^2 \left(\cos^2 \alpha - \sin 2\phi \sin 2\psi \sin^2 \pi \frac{\delta}{\lambda}\right),$$

when applied to biaxal crystals, leads to results analogous to those in Art. 51. When α and γ are such that there are two sets of hyperbolas, the dark arcs of lemniscates on one side of an hyperbolic brush correspond to the bright ones of the lemniscate curves on the other side of it.

Art. 61. When the incident light is circularly polarized, or the emergent light is circularly analyzed, we have to interpret the equation

$$L'^2 = \frac{L^2}{4}\left(1 + \sin 2\psi \cos 2\pi \frac{\delta'}{\lambda}\right),$$

and find, as in uniaxal crystals, sectors in which the dark arcs in one correspond to the bright ones in the contiguous sectors; the separating brushes being of a neutral tint.

Art. 62. When the incident light is circularly polarized and emergent light is circularly analyzed we have to interpret the formula

$$L'^2 = \frac{L^2}{2}\left(1 + \cos 2\pi \frac{\delta''}{\lambda}\right);$$

and there are now no brushes, but the lemniscate curves are seen continuous. In homogeneous light their forms are very accurately shown in experiments.

Art. 63. Prop. *To find the phenomena when a cylindrical rod of unannealed glass is placed between the polarizing and analyzing plates.*

In fig. 57, *ab* being the portion of a cylindrical rod of unannealed glass, set with its axis in the line *SAC*, we have the same expression for the intensity of a beam entering the eye at *e*, as for uniaxal crystals, since the rays are polarized in, and perpendicular to, the planes passing through the axis of the cylinder, as optic axis.

Then in the expression

$$L'^2 = L^2\left(\cos^2\alpha - \sin 2\phi \,.\, \sin 2\psi \sin^2 \pi \frac{\delta}{\lambda}\right),$$

we have the rings and crosses indicated similarly to uniaxal crystals, but δ now depends upon the distance from the axis at which the ray passes through the cylinder, and upon the length of the axis of the cylinder; the beam being considered to pass parallel to the axis and to consist of parallel rays. In this manner the rings are closer as they are more distant from the axis, and also as the cylinder is longer.

Art. 64. In irregular pieces of unannealed glass, and in crystals with irregular crystallization, the retardation is not subject to a simple rule, and the phenomena present many very beautiful but unsymmetrical appearances.

THE END.

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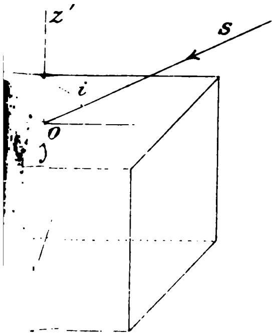

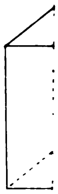

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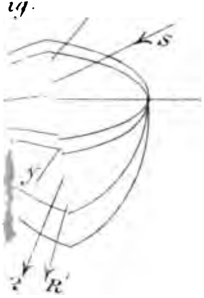

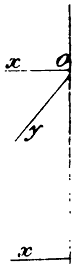

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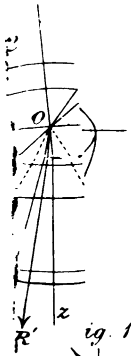

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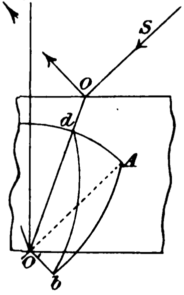

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